

The r-map, c-map and black hole solutions

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by
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Abstract

We consider various geometrical and physical aspects of the r-map and c-map, which are two maps induced by the dimensional reduction of $5d$ and $4d$, $\mathcal{N} = 2$ supergravity coupled to vector multiplets respectively. We treat reduction over a spacelike or timelike dimension on an equal footing, and prove, for the first time, that the target manifold in the image of the timelike c-map is para-quaternion Kähler. In order to do this we provide a new formulation of projective special Kähler geometry based on real Darboux coordinates, which is useful both mathematically and physically in its own right.

As an application we investigate how the r-map and c-map can be used to generate new stationary black hole solutions. In four dimensions we construct new extremal non-BPS solutions, and in both four and five dimensions we construct new non-extremal solutions. We also take the first steps towards constructing new rotating solutions, though at this stage we only recover known solutions.

The systematic and geometrical nature of these constructions allows us to gain a deeper understanding of many familiar properties of black holes in supergravity, such as the attractor mechanism and the transformation of BPS into non-BPS black holes using a field rotation matrix. We also observe an interesting and novel feature relating to non-extremal black holes: in order for solutions to correspond to non-extremal black holes with finite scalar fields we find that the number of integration constants must reduce by half. This suggests that non-extremal black holes always satisfy first order equations similar to their extremal counterparts. For *STU*-like models all calculations are performed explicitly.

Declaration

I hereby declare that all work described in this thesis is the result of my own research unless reference to others is given. None of this material has previously been submitted to this or any other university. All work was carried out in the Theoretical Physics Division of the Department of Mathematical Sciences, University of Liverpool, U.K. during the period of October 2008 until November 2012, with the exception of the period of October 2011 until June 2012 when work was carried out at the Department of Mathematics, Universität Hamburg, Germany.

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Publication list

This thesis contains material that has appeared in the following publications by the author:

- (1) T. Mohaupt and O. Vaughan, “*Non-extremal black holes, harmonic functions, and attractor equations*,” *Class. Quant. Grav.* **27** (2010) 235008 [arXiv:1006.3439].
- (2) T. Mohaupt and O. Vaughan, “*Developments in special geometry*,” *J. Phys. Conf. Ser.* **343** (2012) 012078 [arXiv:1112.2873].
- (3) T. Mohaupt and O. Vaughan, “*The Hesse potential, the c-map and black hole solutions*,” *JHEP* **1207** (2012) 163 [arXiv:1112.2876].
- (4) T. Mohaupt and O. Vaughan, “*Non-extremal black holes from the generalised r-map*,” accepted for publication in *Proceedings of BOSS2011, INFN, Frascati*, May 9-13, 2011. [arXiv:1208.4302].

Unpublished material will also be presented, some of which is scheduled to appear in the following publications:

- (5) V. Cortés, P. A. Dempster, T. Mohaupt and O. Vaughan, “*Special geometry of Euclidean supersymmetry IV: the local c-map*”, to appear.
- (6) T. Mohaupt and O. Vaughan “*Non-extremal black holes and the c-map*”, to appear.

There are also two publications by the author that will not be presented in this thesis, but are worth mentioning since they make extensive use of the results of (3):

- (7) D. Klemm and O. Vaughan, “*Nonextremal black holes in gauged supergravity and the real formulation of special geometry*,” accepted for publication in *JHEP*, [arXiv:1207.2679].
- (8) D. Klemm and O. Vaughan, “*Nonextremal black holes in gauged supergravity and the real formulation of special geometry II*,” [arXiv:1211.1618].

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Chapter 1

Introduction

This thesis is devoted to various physical and geometrical aspects of five- and four-dimensional theories of $\mathcal{N} = 2$ supergravity coupled to vector multiplets. Such theories play an important role in the study of non-perturbative gauge theories [1, 2], string theory compactifications [3, 4, 5], and black hole physics [6, 7, 8]. Theories with extended supersymmetry come equipped with two types of geometry: spacetime geometry and the geometry of the scalar target manifold. For $\mathcal{N} = 2$ supersymmetry the latter are known as ‘special’ geometries and are interesting mathematically in their own right [9, 10, 11].

We will concentrate on two particularly important tools at our disposal: the maps induced by the dimensional reduction of five- or four-dimensional $\mathcal{N} = 2$ supergravity coupled to vector multiplets, which go by the names of the r-map and c-map respectively. By developing our analysis of these maps we gain a deeper understanding of the geometry and physics of these theories as a whole. Moreover, we can use these results to construct new solitonic solutions such as black holes, which play an important role as a testing ground for string theory and other quantum theories of gravity. Let us now give an introduction to, and summary of, the main results:

1. *The real formulation of projective special Kähler geometry*

The standard form of the Lagrangian of $4d$, $\mathcal{N} = 2$ supergravity coupled to vector multiplets is presented in terms of complex scalar fields [12, 13, 14], which reflects the fact that the definition of projective special Kähler geometry is formulated in terms of complex coordinates [13, 11]. However, in certain circumstances it is desirable to use a formulation of this geometry in terms of real coordinates, for instance in the study of non-holomorphic corrections and the OSV conjecture [15, 16, 17, 18]. More recently it has been appreciated that it is also extremely useful for the construction of new black hole solutions [19, 20, 21], which we discuss later. We will also see that the real formulation of projective special Kähler geometry allows us in turn to provide a new formulation of the c-map in terms of real coordinates, and will play an important role in the proof that the target manifold in the image of the temporal c-map is para-quaternion Kähler.

Although the real formulation of affine special Kähler geometry has been known for some time and is relatively straight-forward [11], the real formulation of projective special Kähler geometry has only been considered recently and is technically more challenging. A particular formulation of projective special Kähler geometry in terms of real coordinates has been proposed in [22], though we find there is still room for improvement in various conceptual and practical aspects of this construction. In this

thesis we will present a new formulation of projective special Kähler geometry that is given in terms of real Darboux coordinates and uses horizontal fields on a larger ambient space (the associated conic affine special Kähler manifold). The advantage of this approach is that we expose the underlying conic structure very clearly, and full symplectic covariance is manifest, at least before projecting down. All data is expressed in terms of a single function called the Hesse potential [11], which replaces the role of the holomorphic prepotential when using special real coordinates.

2. The r-map and c-map

The r-map and c-map were first developed around two decades ago [23, 24, 25, 26], and have remained to this day a consistently active area of research for both physicists and mathematicians.

Through dimensional reduction of $5d, \mathcal{N} = 2$ supergravity coupled vector multiplets over a spacelike dimension one obtains a theory of $4d, \mathcal{N} = 2$ supergravity coupled to vector multiplets. This induces a map from the target manifold of the five-dimensional theory, a projective special real manifold, into the target manifold of the four-dimensional theory, a projective special Kähler manifold [24], which is called the r-map. One may alternatively consider the dimensional reduction of the same theory but over a timelike dimension. In this case the four-dimensional theory has an Euclidean spacetime signature and the target manifold is a projective special para-Kähler manifold [27]. We call this the timelike r-map¹ in order to distinguish it from the spacelike r-map.

Recent progress in the r-map has focussed on the generalisation of the spacelike or timelike r-map to include theories that are direct generalisations of $\mathcal{N} = 2$ supergravity coupled to vector multiplets [27, 28], and also to include theories in which $\mathcal{N} > 2$ [29]. Since the timelike generalised r-map involves dimensional reduction over time it can be used as a technique for constructing stationary black hole solutions to supergravity and its generalisations [30, 31, 32]. In this thesis we will provide a review of this generalised r-map based on [27], which we will later use to construct new non-extremal black hole solutions.

The c-map is technically more challenging than the r-map for two reasons. First, the spacelike c-map is a map from a projective special Kähler manifold into a quaternion Kähler manifold [26], which is a relatively difficult type of manifold to study. Second, the isometry group generated by the c-map is more complicated than the r-map [26, 33, 34]. However, for the same reasons the c-map is a particularly interesting construction in both a physical and mathematical sense. We will revisit the original formulation of the spacelike c-map by Ferrara and Sabharwal [26], before providing a new formulation of this map based entirely on real coordinates, which has appeared in the publication [19] by the author. This makes critical use of the real formulation of projective special Kähler geometry discussed previously. We then go on to provide a proof, for the first time, that the target manifold in the image of the timelike c-map is para-quaternion Kähler. This result will appear later in [28].

Using our formalism we also prove the existence of a second integrable and metric compatible complex structure on the quaternion Kähler manifold in the image of the spacelike c-map, in addition to the integrable complex structure found in [35]. Both complex structures are also integrable and metric compatible on the para-quaternion Kähler manifold in the image of the timelike c-map, in which case the second complex structure is also compatible with the para-quaternion structure.

¹In the literature this is also sometimes called the temporal r-map or para-r-map.

3. *Black hole solutions*

Over the years black holes have remained an inexhaustible source of interest to physicists across a broad spectrum of sub-disciplines, such as cosmology, quantum gravity, information theory and recently even condensed matter physics. In this thesis we are interested in black holes from a theoretical point of view, and they will only be considered in isolation from other cosmological objects. The most important feature for us is the existence of an event horizon. We will be particularly interested in static, charged black hole solutions. These come in two varieties: extremal and non-extremal. For charged black holes the mass cannot be lower than the charge $M \geq |Q|$, and we define extremal black holes to be those that saturate this bound, i.e. $M = |Q|$, whereas non-extremal black holes do not, i.e. $M > |Q|$.

In order to give a complete description of black hole physics, up to and including the event horizon, one must appeal to a theory that unifies gravity and quantum mechanics. Conversely, any theory that purports to be a consistent theory of quantum gravity should provide a satisfactory description of black hole physics. This is why black holes play such an important role as a testing ground for string theory and other theories of quantum gravity. Over the past two decades tremendous progress has been made in our understanding of black hole physics in string theory, which is the main motivation for their discussion in this thesis. The matching of the microscopic entropy [6] and macroscopic entropy [7] of a certain class of five-dimensional BPS black holes is one of the most celebrated results of the theory. This discovery ushered in the next two decades of intense interest in black hole physics in string theory and its compactifications to five- or four-dimensional supergravity, which are the theories we will consider in this thesis.

An important subclass of black hole solutions in theories of extended supergravity consists of so-called BPS black holes, which are characterised by preserving some degree of supersymmetry from their parent theory. They saturate a bound relating their mass with the central charge, and in the static case are necessarily extremal. Some well-known black holes can be interpreted as BPS solutions when embedded into supersymmetric theories, such as the extremal Reissner–Nordström black hole [36, 37]. BPS black holes exhibit an interesting feature known as the attractor mechanism, which is a process by which the behaviour of the solution at the horizon is completely by the charges and becomes independent of the asymptotic values of the moduli fields [7, 38, 39, 40]. More generally, BPS solutions can be determined on the whole of spacetime in terms of harmonic functions through so-called generalised stabilisation equations, which reduce to the attractor equations in the near horizon limit [41, 42, 43, 44, 45]. Attractor behaviour can also occur for non-BPS black holes [46, 47], and is now understood to be a consequence of extremality.

The fact that BPS black holes are invariant under certain supersymmetry transformations means that they admit Killing spinors, see e.g. [48, 14], which are the fermionic analogue of Killing vectors. Solving the Killing spinor equations amounts to solving a set of first order differential equations. This is often easier than solving the equations of motion, which are second order. Constructing non-BPS solutions is more challenging since they do not satisfy Killing spinor equations. One approach is to identify ‘fake’ Killing spinor equations, and use these to construct non-BPS solutions in a similar way to their BPS counterparts. However one needs to impose spherical symmetry and/or that the target manifold is a symmetric space [49, 31, 50, 51, 52].

In this thesis we will present a new method of constructing static four-dimensional non-BPS extremal black hole solutions, which has appeared in the publication [19] by

the author. We work directly at the level of the equations of motion and do not assume that spacetime is spherically symmetric or that the target manifold is a symmetric space. While we cannot prove that non-BPS extremal solutions exist for generic models, we will show that such solutions can always be found for a large class of models, which, in particular, contains all models in the image of the r-map. For certain models we construct explicit solutions, such as for *STU*-like models and the $STU + aU^3$ model. For completeness we will also discuss the analogous case in five dimensions, where the procedure for constructing extremal BPS and non-BPS solutions directly from the equations of motion was first pioneered by Mohaupt and Waite in [32].

In contrast to extremal black holes, progress in our understanding of their non-extremal counterparts has been slower and more recent. Since non-extremal black holes cannot be BPS solutions one cannot use the standard method of Killing spinor equations to construct solutions. Various methods have been used to construct non-extremal solutions, which often involve reducing the equations of motion to first order equations (without doubling the total number of equations) [53, 54, 55, 56, 57, 58, 49]. In this case the number of integration constants necessarily reduces by half.

Since the formalism we develop for constructing extremal black holes works directly at the level of equations of motion, it can be adapted in a systematic way to constructing new non-extremal solutions. Since one would not expect the reduced spacetime metric to be flat in the case of non-extremal black holes we have to modify our ansatz from the extremal case. In addition we impose that solutions are spherically symmetric, as one cannot construct static multi-centred non-extremal solutions generically. In both five and four dimensions we find that for *STU*-like models we can integrate the second order equations of motion directly and obtain the general solution. We then identify the subset that correspond to new non-extremal black hole solutions with scalar fields that take finite values. In the five-dimensional case these solutions have appeared in two publications by the author [59, 60], while in the four-dimensional case they are to appear shortly [61].

We also discuss non-extremal black hole solutions to models more general than *STU*-like models, but find that we must set some scalar fields to be proportional to one another. We therefore only obtain solutions with as many independent scalar fields as there are blocks in the metric. This still allows us to show that in five dimensions there is a universal non-extremal solution with constant scalar fields given by the Reissner–Nordström metric, and in the four dimensional class of models mentioned above we find that there is a universal solution with one non-constant scalar field.

A novel property that we observe in all of our non-extremal solutions is that in order for the scalar fields to take finite values on the horizon the number of integration constants must reduce by half. This suggests that non-extremal black hole solutions are always governed by first order equations, just like their extremal counterparts. Although non-extremal solutions have previously been observed to satisfy first order equations, the logic here is different. We start with the general solution to the equations of motion and show that for these solutions to correspond to black holes with finite scalar fields the number of integration constants must reduce by half.

Our formalism also allows one to consider stationary four-dimensional solutions. We consider rotating spacetimes with a flat three-dimensional metric and find that we can very easily recover the known BPS solutions [62, 63, 41, 42]. In the rotating case we observe that non-BPS solutions are characterised by having a non-flat three-dimensional metric, which we will not consider in this thesis. We leave the study of new rotating solutions, both non-BPS and non-extremal, to future work.

This thesis is organised as follows: in chapters 2 and 3 we provide the necessary mathematical and physical background needed for the main part of the thesis. We then follow roughly the chronological order in which the research was undertaken, and which also corresponds to the level of complexity of the material. In chapter 4 we review the generalised r-map, which we then use in chapter 5 to construct new non-extremal black hole solutions in five dimensions. In chapter 6 we present the real formulation of projective special Kähler geometry and perform the c-map, which we then formulate in terms of real coordinates. We recover the results of Ferrara and Sabharwal before providing the proof that the target manifold in the image of the timelike c-map is para-quaternion Kähler. In chapter 7 we use the c-map to construct four-dimensional black hole solutions. We first recover the known rotating and static BPS solutions, before moving on to new static non-BPS and non-extremal solutions. We end with a conclusion and outlook in chapter 8.

1.1 Notation and conventions

We use the conventions of [14]. The signature of the spacetime metric is given by

$$\begin{aligned} (+ + \dots +) & \quad \text{Euclidean spacetime ,} \\ (- + \dots +) & \quad \text{Lorentzian spacetime .} \end{aligned}$$

We will always use the Einstein summation convention for repeated indices unless explicitly stated otherwise, i.e. $A^\mu A_\mu = \sum_\mu A^\mu A_\mu$. We denote the symmetric and antisymmetric parts of a tensor field by

$$A_{(\mu\nu)} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu}) , \quad A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu}) .$$

The coefficients of the Levi-Civita connection D in a coordinate basis are given by

$$D_X Y = X^\mu D_\mu (Y^\lambda \partial_\lambda) = X^\mu \left[\partial_\mu Y^\lambda + \Gamma_{\mu\nu}^\lambda Y^\nu \right] \partial_\lambda ,$$

where

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) .$$

The coefficients of the Riemann curvature tensor are given by

$$R^\rho{}_{\lambda\mu\nu} = \partial_\mu \Gamma^\rho{}_{\nu\lambda} - \partial_\nu \Gamma^\rho{}_{\mu\lambda} + \Gamma^\rho{}_{\mu\sigma} \Gamma^\sigma{}_{\nu\lambda} - \Gamma^\rho{}_{\nu\sigma} \Gamma^\sigma{}_{\mu\lambda} ,$$

which means the curvature two-form is related to the Levi-Civita one-form (spin connection) by

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b .$$

The Ricci curvature is defined by the contraction of the Riemann curvature

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} ,$$

and the scalar curvature is defined by the contraction of the Ricci curvature $R = R^\mu{}_\mu$.

We will frequently switch between a local coordinate basis $\{dx^\mu\}$ of the tangent bundle and a local orthonormal basis $\{e^a\}$, which we refer to as a vielbein basis and the individual e^a as vielbeins. We will use Greek indices for the coordinate basis and Latin indices for vielbein basis

$$g_{\mu\nu} dx^\mu \otimes dx^\nu = \eta_{ab} e^a \otimes e^b .$$

The transformation matrices e_μ^a are defined to satisfy $e^a = e_\mu^a dx^\mu$, and we denote the determinant by

$$\mathbf{e} := \det(e_\mu^a) = \sqrt{|\det(g_{\mu\nu})|}.$$

The m -dimensional epsilon symbol is defined to be

$$\epsilon_{12\dots m} = 1,$$

and the epsilon tensor is given by

$$\varepsilon_{\mu_1\dots\mu_m} = \mathbf{e} \epsilon_{\mu_1\dots\mu_m}.$$

Three useful identities are

$$\varepsilon^{\mu_1\dots\mu_m} = \mathbf{e} \epsilon^{\mu_1\dots\mu_m} = \mathbf{e} g^{\mu_1\nu_1} \dots g^{\mu_m\nu_m} \epsilon_{\nu_1\dots\nu_m} = (-)^f \mathbf{e}^{-1} \epsilon_{\mu_1\dots\mu_m}, \quad (1.1)$$

and

$$\varepsilon_{\mu_1\dots\mu_k\rho_{k+1}\dots\rho_m} \varepsilon^{\nu_1\dots\nu_k\rho_{k+1}\dots\rho_m} = (-)^f p!(m-k)! \delta_{\mu_1}^{\nu_1} \dots \delta_{\mu_k}^{\nu_k}, \quad (1.2)$$

where f is the number of timelike dimensions. It is also useful to note that by definition

$$\mathbf{e} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_m} = \varepsilon^{\mu_1\dots\mu_m} d^m x.$$

We write an arbitrary p -form as

$$\alpha_p = \frac{1}{p!} (\alpha_p)_{\mu_1\dots\mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}.$$

Exterior derivatives act on the left

$$d\alpha_p = \frac{1}{p!} \partial_\nu (\alpha_p)_{\mu_1\dots\mu_p} dx^\nu \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p},$$

as does the interior product $\iota_X g(\cdot, \cdot) = g(X, \cdot)$. The Hodge star is defined by

$$\star \alpha_p = \frac{1}{p!(m-p)!} (\alpha_p)_{\mu_1\dots\mu_p} \varepsilon^{\mu_1\dots\mu_p \nu_{p+1}\dots\nu_m} dx^{\nu_{p+1}} \wedge \dots \wedge dx^{\nu_m}.$$

Consider a p -form α_p in the case where p is even and precisely half the number of dimensions of the manifold. If the metric has Lorentzian signature then $\star\star\alpha_p = -\alpha_p$, and so we can split the p -form into self-dual and anti self-dual parts

$$\alpha_p^\pm = \frac{1}{2} (\alpha_p \mp i\star\alpha_p),$$

where $\star\alpha_p^\pm = \pm i\alpha_p^\pm$. A useful example is the two-form F on a four-dimensional manifold with Lorentzian signature (e.g. spacetime), which we write as

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu.$$

The Hodge-star is given by

$$\star F = \frac{1}{4} F_{\rho\sigma} \varepsilon^{\rho\sigma}{}_{\mu\nu} dx^\mu \wedge dx^\nu,$$

and $\star\star F = -F$. We can therefore split the F into self-dual and anti-self-dual parts

$$F^\pm = \frac{1}{2} (F \mp i\star F).$$

We can write this in components as

$$F_{\mu\nu}^\pm = \frac{1}{2} (F_{\mu\nu} \pm i\tilde{F}_{\mu\nu}),$$

where we have defined

$$\tilde{F}_{\mu\nu} = -\frac{1}{2} F_{\rho\sigma} \varepsilon^{\rho\sigma}{}_{\mu\nu}.$$

Chapter 2

Preliminary mathematics

In this opening chapter we will introduce various mathematical concepts that will be important throughout this thesis. These are based in the field of differential geometry, a firm grasp of which is essential for the study of gravity, gauge theory and supersymmetry. Our aim is to fully understand special geometry, which is needed to define $\mathcal{N} = 2$ supergravity in three, four and five dimensions.

We will begin in section 2.1 with a discussion about connections, which are one of the fundamental objects in differential geometry. This will also serve as a brief introduction to differential geometry, and here we will define many elementary objects. In section 2.2 we will introduce special real manifolds, which are the simplest type of manifolds in special geometry. We then move on to special (para-)Kähler manifolds in section 2.3, and finish with a discussion of (para-)quaternion Kähler manifolds in section 2.4.

2.1 Connections

Connections can be defined on two types of fibre bundles: vector bundles, and principal bundles. In section 2.1.1 we will focus on connections on the tangent bundle, which is a specific type of vector bundle. Here the main motivation is to understand what is meant by an affine connection that is flat and torsion-free (a special connection) or torsion-free and metric compatible (the Levi-Civita connection). This will be important later when we discuss the intrinsic definition of special geometry. In section 2.1.2 we will discuss connections on principal bundles in general. Here the goal is to understand parallel transport and holonomy. An important result is corollary 2.1, which relates the Riemannian holonomy group to the Levi-Civita connection one-form.

This section is based on the well-known texts [64, 65, 66, 67], but for consistency all definitions are based specifically on those in [67]. Throughout this section we will let M denote an arbitrary m -dimensional differentiable manifold.

2.1.1 Connections on the tangent bundle

Definition 2.1 (affine connection). An affine connection ∇ is a map from two smooth vector fields into another smooth vector field¹

$$\begin{aligned}\nabla : \Gamma(TM) \times \Gamma(TM) &\longrightarrow \Gamma(TM) \\ (X, Y) &\longmapsto \nabla_X Y ,\end{aligned}$$

that satisfies the following conditions:

$$\begin{aligned}\nabla_X(Y + Z) &= \nabla_X Y + \nabla_X Z , \\ \nabla_{(X+Y)}Z &= \nabla_X Z + \nabla_Y Z , \\ \nabla_{fX}Y &= f\nabla_X Y , \\ \nabla_X(fY) &= X(f)Y + f\nabla_X Y ,\end{aligned}$$

for all vector fields $X, Y, Z \in \Gamma(TM)$ and smooth functions f on M .

We can write $\nabla_X Y$ in a coordinate basis as

$$\nabla_X Y = X^\mu \nabla_\mu (Y^\lambda \partial_\lambda) = X^\mu \left[\partial_\mu Y^\lambda + \Gamma^\lambda_{\mu\nu} Y^\nu \right] \partial_\lambda , \quad (2.1)$$

where the components $\Gamma^\lambda_{\mu\nu}$ are called the connection coefficients.

An affine connection allows one to define the notion of parallel transport. Consider a curve $c : [a, b] \rightarrow M$, parametrised by t , and a vector field V tangent to the curve. A vector field X is parallel transported along $c(t)$ if

$$\nabla_V X = 0 .$$

Using a curve $c(t)$ and an affine connection ∇ we can compare a vector $X(a)$ at the point $c(a)$ with a vector $X(b)$ at $c(b)$ in a unique and meaningful way. Moreover, we can define a privileged type of curve that has a tangent vector field V which satisfies

$$\nabla_V V = 0 ,$$

or can be brought to this form by a reparametrisation of $c(t)$. Such a curve is called a geodesic.

The action of an affine connection ∇ can be extended to arbitrary tensors by first requiring that the covariant derivative ∇_X acting on a function be the standard directional derivative (which is also the same as the action of the Lie derivative)

$$\nabla_X f = X(f) = \mathcal{L}_X f .$$

This is an extremely versatile formula, from which many properties of the Lie derivative and covariant derivative can be understood. In particular, if ω is a one-form we may require that the covariant derivative satisfies the product rule

$$\nabla_X(\omega(Y)) = X(\omega(Y)) =: (\nabla_X \omega)Y + \omega(\nabla_X Y) ,$$

and so we should define the action of ∇_X on a one-form ω to be

$$\begin{aligned}\nabla_X \omega : \Gamma(TM) &\longrightarrow \mathbb{R} \\ Y &\longmapsto (\nabla_X \omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y) .\end{aligned}$$

¹We denote the set of smooth vector fields by $\Gamma(TM)$, and more generally the set of smooth sections of an arbitrary vector bundle by $\Gamma(V)$.

The action of ∇_X can be extended to an arbitrary tensor field through the rule

$$\nabla_X(T_1 \otimes T_2) = (\nabla_X T_1) \otimes T_2 + T_1 \otimes (\nabla_X T_2) .$$

For a tensor field T of type (r, s) we will often write ∇T to denote the map $\Gamma(TM) \times \Gamma(\otimes^r T^* M \otimes^s TM) \longrightarrow \Gamma(\otimes^r T^* M \otimes^s TM)$ given by $(X, T) \mapsto \nabla_X T$.

Given an affine connection we can define curvature and torsion. The Riemann curvature tensor is defined by

$$\begin{aligned} R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) &\longrightarrow \Gamma(TM) \\ (X, Y, Z) &\longmapsto R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z , \end{aligned}$$

and the torsion tensor by

$$\begin{aligned} T : \Gamma(TM) \times \Gamma(TM) &\longrightarrow \Gamma(TM) \\ (X, Y) &\longmapsto T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] . \end{aligned}$$

A connection is said to be flat if the Riemann curvature vanishes, and torsion-free if the torsion vanishes. Note that these definitions are given in terms of a connection and not any metric that may exist on the manifold.

Let (M, g) be a pseudo-Riemannian manifold with signature (p, q) . An object that will play an important role later on is the connection one-form. In order to define this we must first introduce an orthonormal frame $\{e_a\}$, i.e. a basis of the tangent space that satisfies

$$g(e_a, e_b) = \eta_{ab} , \quad \text{where} \quad \eta = \begin{pmatrix} -\mathbb{1}_p & 0 \\ 0 & \mathbb{1}_q \end{pmatrix} .$$

Any orthonormal frame is related to a coordinate frame $\{\partial_\mu\}$ by an $\text{SL}(m, \mathbb{R})$ transformation denoted e_a^μ

$$e_a = e_a^\mu \frac{\partial}{\partial x^\mu} ,$$

and two orthonormal frames $\{e_a\}$ and $\{e'_a\}$ are related to one another by an $\text{SO}(p, q)$ transformation.

We adopt the physics terminology of referring to $\{e_a\}$ as a vielbein basis, and the individual e_a as vielbeins. We will call the matrices e_a^μ vielbein transformation matrices (or simply transformation matrices). Confusingly, this is different from most of the literature in mathematics, where e_a^μ are referred to as vielbeins and $\{e_a\}$ as an orthonormal frame. The inverse transformation matrices e^a_μ are defined through the expression $e^a_\mu e_b^\mu = \delta_b^a$, and can be written as $e^a_\mu = g_{\mu\nu} \eta^{ab} e_b^\nu$. Just like expression (2.1), we can write $\nabla_X Y$ in a vielbein basis as

$$\nabla_X Y = X^a \nabla_a (Y^c e_c) = X^a \left[e_a Y^c + \gamma_{ab}^c Y^b \right] e_c . \quad (2.2)$$

The connection coefficients in vielbein and coordinate bases are related through

$$\gamma_{ab}^c = e^c_\lambda e_a^\mu (\partial_\mu e_b^\lambda + e_b^\nu \Gamma_{\mu\nu}^\lambda) = e^c_\lambda e_a^\mu \nabla_\mu e_b^\lambda .$$

The vielbein one-forms e^a are defined to be the set of one-forms dual to the vielbeins e_a . We can now define the matrix-valued one-form ω^a_b , called the connection one-form, by

$$\omega^a_b = \gamma^a_{cb} e^c .$$

It will be useful later to interpret ω as a one-form that takes values in the Lie algebra of the general linear group $\omega \in \mathfrak{gl}(m, \mathbb{R}) \otimes T^*M$ (note that actually every real $m \times m$ matrix is an element of $\mathfrak{gl}(m, \mathbb{R})$). The connection one-form satisfies Cartan's structure equations

$$\begin{aligned} T^a &= de^a + \omega^a_b \wedge e^b, \\ R^a_b &= d\omega^a_b + \omega^a_c \wedge \omega^c_b, \end{aligned}$$

where $T^a = \frac{1}{2}T^a_{bc}e^b \wedge e^c$ is the torsion two-form and $R^a_b = \frac{1}{2}R^a_{bcd}e^c \wedge e^d$ is the curvature two-form. These are related to the previously defined torsion and Riemann curvature through

$$\begin{aligned} T^a_{bc} &= \langle e^a, T(e_b, e_c) \rangle, \\ R^a_{bcd} &= \langle e^a, R(e_b, e_c, e_d) \rangle. \end{aligned}$$

Definition 2.2 (Levi-Civita connection). Given a pseudo-Riemannian manifold (M, g) the Levi-Civita connection, denoted D , is the unique connection that is metric compatible and torsion-free

$$Dg = 0, \quad T(X, Y) = 0,$$

for all $X, Y \in \Gamma(TM)$.

We can formulate the definition of the Levi-Civita connection in terms of a connection one-form by the following proposition:

Proposition 2.1 . The Levi-Civita one-form is the unique connection one-form that satisfies

- (i) $\omega^a_b \in \mathfrak{so}(p, q)$,
- (ii) $0 = de^a + \omega^a_b \wedge e^b$,

where (i) corresponds to metric compatibility and (ii) corresponds to being torsion-free.

Proof. (Nakahara) By Cartan's structure equations it follows that the torsion-free condition implies (ii). We need to show that metric compatibility implies (i). First define

$$\gamma_{abc} = \eta_{ad}\gamma^d_{bc}, \quad \omega_{ab} = \eta_{ad}\omega^d_b.$$

Since $\nabla_\mu g = 0$ for metric compatible connections we have

$$\begin{aligned} \gamma_{abc} &= \eta_{ad}e^d_\lambda e_b^\mu \nabla_\mu e_c^\lambda = \eta_{ad}e_b^\mu \nabla_\mu (e^d_\lambda e_c^\lambda) - \eta_{ad}e_c^\lambda e_b^\mu \nabla_\mu e^d_\lambda \\ &= \eta_{cd}e^d_\lambda e_b^\mu \nabla_\mu e_a^\lambda = -\gamma_{cba}, \end{aligned}$$

and since $\omega_{ac} = \gamma_{abc}e^b$ we have

$$\eta_{ac}\omega^c_b = -\eta_{bc}\omega^c_a,$$

which proves (i). Since the Levi-Civita connection is unique the connection coefficients γ^a_{bc} are unique (up to $SO(p, q)$ rotations of the vielbein basis), and, hence, the Levi-Civita one-form is unique. \square

2.1.2 Connections on principal bundles

Definition 2.3 (principal bundle). A principal bundle $P(M, G)$ is a fibre bundle $P \xrightarrow{\pi} M$ in which the group G acts freely and transitively on the fibres.

For any fibre bundle the transition functions act on a fibre on the left

$$L_g u = gu ,$$

where $\pi(u) = \pi(gu) = p$. For a principal bundle there is an additional group action, which acts on a fibre on the right

$$R_g u = ug ,$$

where again $\pi(u) = \pi(ug) = p$. By definition the group action is free and transitive on each fibre $\pi^{-1}(p)$, which means that any two points $u, v \in \pi^{-1}(p)$ are related by a unique element $g \in G$ through the right action $R_g u = v$. The right and left actions commute.

Example 2.1 (frame bundle). A particularly useful example of a principal bundle is the frame bundle. A frame $\{\theta_\alpha\}$ at $p \in M$ is simply a basis of the tangent space $T_p M$. Since any two frames are related by a $\text{GL}(m, \mathbb{R})$ transformation, the frame bundle is a $\text{GL}(m, \mathbb{R})$ principal bundle. Moreover, since $\text{GL}(m, \mathbb{R})$ is also the structure group of the tangent bundle, the frame bundle is the principal bundle associated with the tangent bundle. The group action of $a \in \text{GL}(m, \mathbb{R})$ on a frame $\{\theta_\alpha\}$ is defined by the by the right action

$$R_a \theta_\alpha = \theta_\beta a^\beta{}_\alpha .$$

On each chart (U_i, x^μ) and (U_j, y^ν) we can write

$$\theta_\alpha \Big|_{\pi^{-1}(U_i)} = (\theta_i)_\alpha{}^\mu \frac{\partial}{\partial x^\mu} , \quad \theta_\alpha \Big|_{\pi^{-1}(U_j)} = (\theta_j)_\alpha{}^\nu \frac{\partial}{\partial y^\nu} ,$$

where $(\theta_i), (\theta_j) \in \text{GL}(m, \mathbb{R})$. Local trivialisations on U_i and U_j are therefore given by

$$\phi_{i,p}^{-1}(u) = (p, (\theta_i)_\alpha{}^\mu) , \quad \phi_{j,p}^{-1}(u) = (p, (\theta_j)_\alpha{}^\nu) .$$

On overlapping charts we have $\theta_\alpha \Big|_{\pi^{-1}(U_i)} = \theta_\alpha \Big|_{\pi^{-1}(U_j)}$ and, hence,

$$(\theta_i)_\alpha{}^\mu = \frac{\partial x^\mu}{\partial y^\nu} (\theta_j)_\alpha{}^\nu .$$

The transition functions must satisfy $(\theta_i)_\alpha{}^\mu = (t_{ij})^\mu{}_\nu (\theta_j)_\alpha{}^\nu$ and are therefore given by

$$(t_{ij})^\mu{}_\nu = \frac{\partial x^\mu}{\partial y^\nu} \in \text{GL}(m, \mathbb{R}) ,$$

which confirms that the structure group is $\text{GL}(m, \mathbb{R})$.

The vertical subspace $V_u P \subset T_u P$ of a principal bundle is defined to be the subspace tangent to the fibre $\pi^{-1}(p)$, and is given by the kernel of π_*

$$Y \in V_u P \quad \Leftrightarrow \quad \pi_* Y = 0 .$$

There is an isomorphism from the Lie algebra of G to the vertical subspace given by

$$\begin{aligned} \sharp : \mathfrak{g} &\longrightarrow V_u P \\ A &\longmapsto A_u^\sharp, \end{aligned}$$

where

$$A_u^\sharp(f) = \left. \frac{d}{dt} f(u \exp(tA)) \right|_{t=0}.$$

The vector field A^\sharp is called the fundamental vector field generated by A . A complement to the vertical subspace is called a horizontal subspace, which is not unique in general. A choice of unique horizontal subspace at every point $u \in P$ is equivalent to choosing a connection

$$\text{unique horizontal subspace} \quad \Leftrightarrow \quad \text{connection}$$

Definition 2.4 (connection on a principal bundle). A connection ∇ on a principal bundle $P(M, G)$ is the unique separation of the tangent space at every point $u \in P$ into a vertical subspace $V_u P$ and horizontal subspace $H_u^\nabla P$ such that

- (i) $T_u P = V_u P \oplus H_u^\nabla P$.
- (ii) Any smooth vector field X on P separates into $X = X^V + X^H$, where X^V and X^H are smooth vector fields and $X_u^V \in V_u P$ and $X_u^H \in H_u^\nabla P$ for every point $u \in P$.
- (iii) $H_{ug}^\nabla P = R_{g*} H_u^\nabla P$ for any $g \in G$.

Just like for an affine connection on the tangent space, the concept of a connection on a principal bundle can be re-expressed in terms of a one-form. On a principal bundle we define a connection one-form $\omega \in \mathfrak{g} \otimes T^*P$ to be the projection of the tangent space onto the vertical component, which is isomorphic to $\mathfrak{g} \simeq V_u P$, that satisfies

- (i) $\omega(A^\sharp) = A$, $A \in \mathfrak{g}$,
- (ii) $R_g^* \omega = \text{Ad}_{g^{-1}} \omega$.

A horizontal subspace is then defined by the kernel of ω

$$X \in H_u^\nabla P \quad \Leftrightarrow \quad \omega(X) = 0,$$

and it is clear that a connection ∇ is completely equivalent to a connection one-form ω .

Given a coordinate patch U_i of M and local section σ_i of P , we can locally define a \mathfrak{g} -valued one-form \mathcal{A}_i on U_i given by

$$\mathcal{A}_i = \sigma_i^* \omega \in \mathfrak{g} \otimes T^*U_i. \quad (2.3)$$

On the overlap of patches U_i and U_j these one-forms are related by

$$\mathcal{A}_j = t_{ij}^{-1} \mathcal{A}_i t_{ij} + t_{ij}^{-1} dt_{ij}.$$

Conversely, if we have a \mathfrak{g} -valued one-form \mathcal{A}_i and section σ_i on each patch U_i of an open cover of M , and the \mathcal{A}_i satisfy the above relation on the overlap of patches, then there exists a unique connection one-form ω on M such that $\mathcal{A}_i = \sigma_i^* \omega$ on each patch.

Given a principal bundle one may consider a so-called associated vector bundle, which we denote by $P \times_{\rho} V \xrightarrow{\pi_E} M$ where ρ is a k -dimensional representation of G and V is a vector space of dimension k . This is defined by identifying points (u, v) with $(ug, \rho(g)^{-1}v)$, where $u \in P$, $v \in V$, $g \in G$. The fibre structure is given by $\pi_E(u, v) = \pi_E(ug, \rho(g)^{-1}v) = \pi(u)$, and local trivialisations by $\psi_i : U_i \times V \rightarrow \pi_E^{-1}(U_i)$. Consider a principal bundle with $G \subset \text{GL}(m, \mathbb{R})$ and an associated rank m vector bundle in the standard representation of $\text{GL}(m, \mathbb{R})$. In this case there is a 1:1 relationship between connections on the principal bundle and connections on the associated vector bundle.

Example 2.2 (frame and tangent bundles). The tangent bundle is a rank m vector bundle associated with the frame bundle $\text{GL}(m, \mathbb{R})$. The transition functions act on the fibre on the left through the standard representation of $\text{GL}(m, \mathbb{R})$, and there is therefore a 1:1 correspondence between connections on the two bundles. In particular, the Levi-Civita connection is uniquely defined on the tangent bundle and therefore also on the frame bundle.

Definition 2.5 (horizontal lift). Let $P(M, G)$ be a principal bundle and $\gamma : [0, 1] \rightarrow M$ be a curve in M . A curve $\tilde{\gamma} : [0, 1] \rightarrow P$ in P is a horizontal lift of γ if $\pi \circ \tilde{\gamma} = \gamma$ and the vector field tangent to $\tilde{\gamma}(t)$ always lies in $H_{\tilde{\gamma}(t)}^{\nabla} P$.

Proposition 2.2 . For a curve $\gamma : [0, 1] \rightarrow M$ and point $u \in \pi^{-1}(\gamma(0))$ there exists a unique horizontal lift $\tilde{\gamma}$ such that $\tilde{\gamma}(0) = u$.

Proof. (Nakahara) Consider a patch U_i , section σ_i and a curve $\tilde{\gamma}(t)$ in P such that $\pi \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}(0) = u$. Such a curve can always be written in $\pi^{-1}(U_i)$ as

$$\tilde{\gamma}(t) = \sigma_i(t)g_i(t), \quad (2.4)$$

where we have introduced the notation $\sigma_i(\gamma(t)) = \sigma_i(t)$. Note that $g_i(t)$ is a G -valued function of the curve parameter and not of a point in P . Let X denote the vector field tangent to γ , and \tilde{X} the vector field tangent to $\tilde{\gamma}$. We will employ the standard notation

$$\tilde{X}_u = \left. \frac{d}{dt} \tilde{\gamma}(t) \right|_{t=0}, \quad (2.5)$$

to represent the vector which acts on a smooth function f on P as

$$\tilde{X}_u(f) = \left. \frac{d}{dt} f(\tilde{\gamma}(t)) \right|_{t=0}.$$

Using the decomposition (2.4) the vector \tilde{X}_p can be expanded as

$$\tilde{X}_u = \underbrace{\left. \frac{d}{dt} \sigma_i(t) \right|_{t=0} g_i(0)}_1 + \underbrace{\sigma_i(0) \left. \frac{d}{dt} g_i(t) \right|_{t=0}}_2. \quad (2.6)$$

Since

$$\begin{aligned} (R_{g_i(0)*}(\sigma_{i*}X))_u(f) &= \left. \frac{d}{dt} f(R_{g_i(0)}\sigma_i(t)) \right|_{t=0} \\ &= \left. \frac{d}{dt} f(\sigma_i(t)g_i(0)) \right|_{t=0}, \end{aligned}$$

we can write the first term in (2.6) as

$$\mathbf{1} = R_{g_i(0)*}(\sigma_{i*}X)_u.$$

To analyse the second term we may use the decomposition (2.4) to write

$$\sigma_i(0) \frac{d}{dt} g_i(t) \Big|_{t=0} = \tilde{\gamma}(0) g_i^{-1}(0) \frac{d}{dt} g_i(t) \Big|_{t=0} .$$

Observe that

$$\begin{aligned} g_i^{-1}(0) dg_i(X) \Big|_{t=0} &= g_i^{-1}(0) \frac{d}{dt} g_i(t) \Big|_{t=0} \\ &= \frac{d}{dt} [g_i^{-1}(0) g_i(t)] \Big|_{t=0} \in T_e G \simeq \mathfrak{g} , \end{aligned}$$

since $g_i^{-1}(0) g_i(t) = e$ at $t = 0$. Therefore $\tilde{\gamma}(0) g_i^{-1}(0) \frac{d}{dt} g_i(t) \Big|_{t=0}$ represents the vertical vector field $(g_i(t) dg_i(X) \Big|_{t=0})^\sharp$ at $\tilde{\gamma}(0) = u$. We can then write the second term in (2.6) as

$$\mathbf{2} = (g_i(t) dg_i(X))_u^\sharp .$$

We can therefore write (2.6) as

$$\tilde{X}_u = R_{g_i(0)*}(\sigma_{i*} X)_u + (g_i^{-1}(0) dg_i(X))_u^\sharp ,$$

which we can write at an arbitrary point on the curve $\tilde{\gamma}(t)$ as

$$\tilde{X} = R_{g_i(t)*}(\sigma_{i*} X) + (g_i(t)^{-1} dg_i(X))^\sharp . \quad (2.7)$$

Note that the first term lies in $H_{\tilde{\gamma}(t)}^\nabla P$ and the second term in $V_{\tilde{\gamma}(t)} P$.

The vector field \tilde{X} tangent to $\tilde{\gamma}$ is the horizontal lift of γ if it satisfies $\omega(\tilde{X}) = 0$. Acting with ω on (2.7) we have

$$\begin{aligned} 0 = \omega(\tilde{X}) &= R_{g_i(t)*} \omega(\sigma_{i*} X) + g_i^{-1}(t) dg_i(X) \\ &= g^{-1}(t) \omega(\sigma_{i*} X) g(t) + g_i^{-1}(t) dg_i(X) . \end{aligned}$$

We can act on the left with $g(t)$ to obtain

$$\frac{dg_i(t)}{dt} = -\omega(\sigma_{i*} X) g_i(t) ,$$

and by the definition of \mathcal{A}_i in (2.3) we can write this as

$$\frac{dg_i(t)}{dt} = -\mathcal{A}_i(X) g_i(t) . \quad (2.8)$$

This is simply an ODE with initial condition $u = \sigma_i(0) g_i(0)$, and by the fundamental theorem of ODEs (the Picard–Lindelöf theorem) a unique solution is guaranteed to exist. \square

For a point $u \in P$ with $\pi(u) = p$, consider a path $\gamma : [0, 1] \rightarrow M$ that satisfies $\gamma(0) = \gamma(1) = p$. Such a path is called a loop, and we denote the set of loops at p by

$$C_p = \{ \gamma : [0, 1] \rightarrow M \mid \gamma(0) = \gamma(1) = p \} .$$

The horizontal lift of a loop defines a transformation between points on the fibre $P_\gamma : \pi^{-1}(p) \rightarrow \pi^{-1}(p)$, which is compatible with the right action of the group

$$P_\gamma(ug) = P_\gamma(u)g .$$

The point $u_1 = \tilde{\gamma}(1) = P_\gamma(u) \in \pi^{-1}(p)$ is known as the parallel transport of u along the loop γ .

Definition 2.6 (holonomy group). The set of group transformations induced by parallel transport of a point $u \in P$ around all loops $\gamma \in C_p(M)$ is given by

$$\text{Hol}_u(\nabla) = \{g \in G \mid P_\gamma(u) = ug, \text{ for some } \gamma \in C_p(M)\} \subset G ,$$

and is known as the holonomy group at u .

Throughout this thesis we will assume that all manifolds are connected, in which case the holonomy group at every point $u \in P$ is isomorphic to the holonomy group at every other point. One can then talk about the holonomy group of a connection on a principal bundle $\text{Hol}(\nabla) \subset G$ without reference to a particular point. We denote the Lie algebra of the holonomy group as $\text{hol}(\nabla) \subset \mathfrak{g}$. Whenever a manifold is not simply-connected we will take the holonomy group to be the restricted holonomy group, which is generated by the set of all loops that can be shrunk to a point without any topological obstructions.

We will now present a theorem that relates the holonomy group to the connection one-form. This plays an important role later on in this thesis. We will give two different proofs: first a constructive proof that is valid only on manifolds which admit a global coordinate system. The second proof makes use of the Ambrose–Singer theorem and is valid for all manifolds.

Theorem 2.1 . Let $P(M, G)$ be a principal bundle over a connected manifold M equipped with a connection ∇ . If the connection one-form takes values in \mathfrak{h} , a Lie subalgebra of \mathfrak{g} , then the holonomy group is a subgroup of $H \subset G$, i.e.

$$\omega \in \mathfrak{h} \otimes T^*M , \quad \mathfrak{h} \subset \mathfrak{g} \quad \Rightarrow \quad \text{Hol}(\nabla) \subset H .$$

Proof (1). Suppose the manifold can be covered by a single coordinate patch U_1 , and that the Lie-algebra valued one-form \mathcal{A}_1 takes values in \mathfrak{h}

$$\mathcal{A}_1 = \sigma_1^* \omega \in \mathfrak{h} \otimes T^*U_1 .$$

The expression (2.8) is valid for any point on a curve in the coordinate patch U_1 and therefore M . The holonomy group consists of all group elements $g_1(1)$ of the form $\tilde{\gamma}(0) = \tilde{\gamma}(1)g_1(1)$ where the curve γ is a loop. Since $g_1(t)$ must satisfy (2.8) for all t we have

$$g_1(1) \in \text{Hol}(\nabla) \quad \Leftrightarrow \quad \left. \frac{dg_1}{dt} \right|_{t=1} = -\mathcal{A}_1(X)g_1(1) .$$

Since $\mathcal{A}_1(X) \in \mathfrak{h}$ and $g_1(0) = e$ it follows that $g_1(1) \in H$. □

Proof (2). The curvature two-form of a connection on a principal bundle is defined to be

$$\Omega = \nabla \omega \in \mathfrak{g} \otimes \Lambda^2(P) .$$

This satisfies Cartan's structure equations

$$\Omega = d\omega + \omega \wedge \omega ,$$

which generalise Cartan's structure equations for an affine connection. We now need to make use of the following celebrated theorem by Ambrose and Singer:

Lemma 2.1 (Ambrose–Singer theorem). Let $P(M, G)$ be a principal bundle with a connection ∇ . Then the Lie algebra of the holonomy group at the point u is given by

$$\text{hol}_u(\nabla) = \{ \Omega_u(X, Y) = -\omega_u([X, Y]) \mid X, Y \in H_u P \} .$$

See [64] for proof.

From this lemma it follows immediately that if $\omega \in \mathfrak{h} \subset \mathfrak{g}$ then $\text{Hol}(\nabla) \subset H$, which completes proof (2). □

Corollary 2.1 . If the Levi-Civita connection one-form of a pseudo-Riemannian manifold M takes values in the Lie subalgebra $\mathfrak{h} \subset \mathfrak{so}(p, q)$ then the Riemannian holonomy group is given by

$$\text{Hol}(D) \subset H .$$

2.2 Special real manifolds

Special real manifolds come in two types: affine special real and projective special real². These correspond to the target manifolds of $5d, \mathcal{N} = 2$ rigid vector multiplets and $5d, \mathcal{N} = 2$ vector multiplets coupled to supergravity respectively [68, 69]. However, we will later consider generalisations of $5d, \mathcal{N} = 2$ vector multiplets coupled to supergravity, and accordingly we need to consider a more general type of geometry, which we will simply call generalised projective special real geometry.

We will follow the procedure outlined in [59] for defining a generalised projective special real manifold, which itself was based on [34] and unpublished work by Vicente Cortés and Thomas Mohaupt. The method is to first define a \mathfrak{d} -conic Hessian manifold. A generalised projective special real manifold is then defined as a particular hypersurface in, or equivalently quotient of, a \mathfrak{d} -conic Hessian manifold.

Definition 2.7 (Hessian manifold). A pseudo-Riemannian manifold (M, g) is Hessian if there exists a flat, torsion-free connection ∇ such that ∇g is completely symmetric.

For a flat, torsion-free connection one may cover the manifold M with a set of normal coordinate charts, in which the coordinates h^I are flat with respect to the connection, i.e.

$$\nabla dh^I = 0 \quad \Rightarrow \quad \nabla_X Y = X^I (\partial_I Y^J) \partial_J , \quad \text{where} \quad \partial_I = \frac{\partial}{\partial h^I} .$$

In such coordinates the requirement that ∇g is completely symmetric is equivalent to

$$\partial_I g_{JK} = \partial_J g_{IK} ,$$

where $g_{IJ} = g(\partial_I, \partial_J)$. This is the condition that the metric g is locally given by the second derivatives of a function [70]

$$g_{IJ} = \frac{\partial^2}{\partial h^I \partial h^J} H .$$

We call the function H the Hesse potential, and it is unique up to terms linear in h^I .

²Often the prefix ‘very’ is also added. This sentence would then read: ‘affine very special real and projective very special real’.

Definition 2.8 (affine special real). An affine special real manifold (M, g, ∇) is a pseudo-Riemannian manifold (M, g) equipped with a flat, torsion-free ‘special’ connection ∇ such that ∇g is completely symmetric and the corresponding Hesse potential is a cubic polynomial.

Definition 2.9 (d-conic Hessian). A d-conic Hessian manifold (M, g, ∇, ξ) is a pseudo-Riemannian manifold (M, g) equipped with a flat, torsion-free ‘special’ connection ∇ and vector field ξ such that

- (i) ∇g is completely symmetric.
- (ii) $D\xi = \frac{d}{2}\mathbb{1}$, where D is the Levi-Civita connection.
- (iii) $\nabla\xi = \mathbb{1}$.

Let us discuss each condition in turn. Condition (i) tells us that the manifold is Hessian with respect to the special connection. We call the normal coordinates h^I ‘special’ coordinates as they are associated with the special connection.

To analyse condition (ii) we follow a similar procedure to [71], which deals with the specific case $d = 2$. We begin by considering the Koszul formula

$$2g(D_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ + g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X) .$$

The part symmetric in X and Z gives

$$g(D_X Y, Z) + g(D_Z Y, X) = Yg(X, Z) + g([X, Y], Z) - g([Y, Z], X) \\ = (\mathcal{L}_Y g)(X, Z) . \quad (2.9)$$

Plugging in $Y = \xi$ we find

$$\mathcal{L}_\xi g = dg ,$$

which in component form reads

$$\mathcal{L}_\xi g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho} \partial_\nu \xi^\rho + g_{\nu\rho} \partial_\mu \xi^\rho = dg_{\mu\nu} . \quad (2.10)$$

This means that ξ is a homothetic Killing vector field of weight d . Condition (ii) also implies that

$$D_\mu \xi_\nu = D_\nu \xi_\mu \quad \Rightarrow \quad \xi_\mu = \partial_\mu f ,$$

for some function f , and the metric can be written as

$$g_{\mu\nu} = \frac{d}{2} D_\mu \partial_\nu f .$$

We now define a new function V by

$$V := g^{\mu\nu} \partial_\mu f \partial_\nu f = g(\xi, \xi) .$$

Taking derivatives we find

$$\partial_\mu V = d\partial_\mu f ,$$

and we can make the choice $V = df = g(\xi, \xi)$. We now choose f to be one of our coordinates $x^0 = f$, and so

$$\xi = df \frac{\partial}{\partial f} \quad \Rightarrow \quad \xi^\mu = df \delta_0^\mu .$$

We can deduce the g_{00} component of the metric through

$$V = g_{\mu\nu}\xi^\mu\xi^\nu = g_{00}(\mathbf{d}f)^2 \quad \Rightarrow \quad g_{00} = \frac{1}{\mathbf{d}f} ,$$

and since ξ is orthogonal to surfaces of constant f there can be no cross terms in the metric $g_{0i} = 0$. We can therefore write the metric as

$$g = \frac{df^2}{\mathbf{d}f} + h_{ij}(f, x^k)dx^i dx^j .$$

We now define the new radial coordinate r such that $r^{\mathbf{d}} = \mathbf{d}f$, and we have

$$\xi = r \frac{\partial}{\partial r} .$$

From the (i, j) component of (2.10) we have

$$r \frac{\partial}{\partial r} h_{ij}(r, x^k) = \mathbf{d}h_{ij}(r, x^k) \quad \Rightarrow \quad h_{ij}(r, x^k) = r^{\mathbf{d}} \bar{g}_{ij}(x^k) ,$$

and the metric therefore decomposes as

$$g = r^{\mathbf{d}-2} dr^2 + r^{\mathbf{d}} \bar{g}_{ij}(x) dx^i dx^j . \quad (2.11)$$

One can define new set of coordinates $y^I = (r, rx^i)$, for which the homothetic Killing vector ξ becomes an Euler vector field

$$\xi = y^I \frac{\partial}{\partial y^I} .$$

In these coordinates

$$\left[\xi, \frac{\partial}{\partial y^I} \right] = - \frac{\partial}{\partial y^I} . \quad (2.12)$$

We now plug in $X = \frac{\partial}{\partial y^I}, Y = \xi, Z = \frac{\partial}{\partial y^J}$ to the symmetric part of the Koszul formula (2.9) and make use of (2.12) to find

$$\xi g_{IJ}(y) = (\mathbf{d} - 2) g_{IJ}(y) .$$

This shows that condition (ii) can be seen as a homogeneity condition on the components of the metric (in particular coordinates).

The last condition (iii) ensures that ξ is the Euler field associated with, in particular, the special coordinates h^I

$$\xi = h^I \frac{\partial}{\partial h^I} .$$

Therefore the components of the metric in a basis of special coordinates are homogeneous functions of degree $(\mathbf{d} - 2)$

$$\xi g_{IJ}(h) = (\mathbf{d} - 2) g_{IJ}(h) .$$

Contracting this with h^I, h^J we have

$$g_{IJ}(h) = \frac{\partial^2}{\partial h^I \partial h^J} \left[\frac{1}{\mathbf{d}(\mathbf{d} - 1)} (g_{IJ} h^I h^J) \right] ,$$

and we see that one can always choose a unique Hesse potential that is a homogeneous function of degree \mathfrak{d} , given by

$$H = \frac{1}{\mathfrak{d}(\mathfrak{d} - 1)} g_{IJ} h^I h^J .$$

Let us now introduce a second metric on M , given by

$$a = \left(\frac{\partial^2}{\partial h^I \partial h^J} \tilde{H} \right) dh^I \otimes dh^J ,$$

where we have defined

$$\tilde{H} := -\frac{1}{\mathfrak{d}} \log H .$$

We can write this metric in a special coordinate basis as

$$a_{IJ} = -\frac{1}{\mathfrak{d}} \left(\frac{H_{IJ}}{H} - \frac{H_I H_J}{H^2} \right) , \quad (2.13)$$

where H_I and H_{IJ} are the first and second derivatives of the Hesse potential respectively. If the metric g has signature $(+ - \dots -)$, which is positive in the ξ direction (this is the case in $5D$, $\mathcal{N} = 2$ supergravity), then a is strictly positive definite $(+ + \dots +)$. The vector field ξ is now a Killing vector field of a

$$\mathcal{L}_\xi a = 0 .$$

Definition 2.10 (generalised projective special real). A generalised special real manifold (\bar{M}, \bar{g}) is a hypersurface of constant H in a \mathfrak{d} -conic Hessian manifold, with metric induced from a .

It is particularly convenient to consider the hypersurface defined by $H = 1$

$$\bar{M} \simeq \{H = 1\} \subset M .$$

Let us denote the embedding of \bar{M} into M given by this hypersurface $H = 1$ by $i : \bar{M} \rightarrow M$. For this embedding both the pull-back of $-\frac{1}{\mathfrak{d}}g$ and a induce the same metric on \bar{M}

$$\bar{g} = i^* \left(-\frac{1}{\mathfrak{d}} \partial^2 H \right) = i^* \left(\partial^2 \tilde{H} \right) .$$

Let ϕ^x denote local coordinates on \bar{M} , which therefore parametrise the hypersurface $H = 1$. The metric in these coordinates is written as

$$\bar{g} = \bar{g}_{xy} d\phi^x \otimes d\phi^y = \left(a_{IJ} \frac{\partial h^I}{\partial \phi^x} \frac{\partial h^J}{\partial \phi^y} \right) \Big|_{H=1} d\phi^x \otimes d\phi^y .$$

A particularly useful set of coordinates is given by

$$\phi^x = \frac{h^x}{h^0} , \quad h^0 = \hat{H}(\phi^1, \dots, \phi^n)^{-\frac{1}{\mathfrak{d}}} := H \left(1, \frac{h^1}{h^0}, \dots, \frac{h^n}{h^0} \right)^{-\frac{1}{\mathfrak{d}}} . \quad (2.14)$$

It is worth noting that one can also realise \bar{M} as the quotient manifold $M/\mathbb{R}^{>0}$ with quotient metric obtained from (M, a) .

For the special case that $\mathfrak{d} = 3$ and the Hesse potential is a polynomial then (\bar{M}, \bar{g}) represents the target manifold of $5D$, $\mathcal{N} = 2$ supergravity coupled to vector multiplets [69]. The matrix a_{IJ} restricted to the hypersurface $H = 1$ provides the coupling matrix for the kinetic term of the gauge fields.

2.3 Special (para-)Kähler manifolds

In this section we will introduce the notion of a special-Kähler manifold. Like special real manifolds these come in two types: affine special Kähler manifolds and projective special Kähler manifolds, which correspond to the target manifolds of $4d, \mathcal{N} = 2$ rigid vector multiplets and $4d, \mathcal{N} = 2$ vector multiplets coupled to supergravity respectively [12, 13]. We will also need to consider the Euclidean versions of these theories, which have slightly different target space geometries. The target manifold geometry of Euclidean $4d, \mathcal{N} = 2$ rigid vector multiplets and $4d, \mathcal{N} = 2$ vector multiplets coupled to supergravity are given by affine special para-Kähler manifolds and projective special para-Kähler manifolds respectively [72, 27]. The prefix ‘para’ means that we replace the underlying complex manifold with a so-called para-complex (or split-complex) manifold. In a para-complex manifold the original role of the imaginary unit i , which squares to -1 , is replaced by a para-imaginary unit e , which squares to $+1$, while the principle of conjugation is still respected

$$i^2 = -1, \quad \bar{i} = -i \quad \longrightarrow \quad e^2 = +1, \quad \bar{e} = -e.$$

This definition will be made precise shortly. From a para-complex manifold one can correspondingly define a para-Kähler manifold and a special para-Kähler manifold. Since the difference between complex manifolds and para-complex manifolds is encoded completely in a particular choice of sign, we can actually describe both manifolds simultaneously by introducing ‘ ϵ -notation’, where

$$\epsilon = \begin{cases} -1, & \text{complex manifold,} \\ +1, & \text{para-complex manifold.} \end{cases}$$

Accordingly we define the ϵ -imaginary unit i_ϵ to satisfy

$$1.i_\epsilon = i_\epsilon.1 = i_\epsilon, \quad i_\epsilon^2 = \epsilon. \quad (2.15)$$

The definition of an ϵ -complex number can be made precise by formally adjoining i_ϵ to the field of real numbers $\mathbb{R}[i_\epsilon] = \mathbb{C}_\epsilon$ such that i_ϵ satisfies (2.15). An ϵ -complex manifold is therefore a complex manifold when $\epsilon = -1$, and a para-complex manifold when $\epsilon = +1$, likewise for ϵ -Kähler and special ϵ -Kähler.

The precise definition of affine special Kähler and projective special Kähler manifolds has undergone many refinements over the years [73, 74, 75]. We will use the definition of affine special Kähler found in [11], which gives a mathematically rigorous and intrinsic definition of affine special Kähler geometry³. We will use the definition of a projective special Kähler manifold found in [27], which uses the Kähler quotient of a particular type of affine special Kähler manifold that has the property of being conical. At the time of writing no intrinsic definition of projective special Kähler geometry exists. Also, the definition of affine special Kähler given in [11] can easily be adapted to give an affine special para-Kähler version [72]. The definition of projective special Kähler found in [27] already contains the definition of the para case.

In the previous section we found that we could generalise the definition of projective special real manifolds in a meaningful way. However, at the time of writing no satisfactory analogous generalisation of projective special Kähler geometry exists,

³By intrinsic we mean a description in terms of data on the tangent bundle and its associated bundles, which are constructed directly from the coordinate charts.

though some clues have been uncovered by considering the dimensional reduction of five-dimensional theories with generalised projective special real geometry [34]. The search for a meaningful generalisation of projective special Kähler geometry remains an interesting open question.

Later in the thesis we will need to prove that certain manifolds are ϵ -complex by the integrability of almost ϵ -complex structures. It is therefore necessary to go through the definition of ϵ -complex manifolds from the bottom up. After that we can introduce ϵ -Kähler manifolds, and finally special ϵ -Kähler manifolds.

Definition 2.11 (almost ϵ -complex manifold). An almost ϵ -complex manifold (M, J) is a manifold endowed with a globally defined smooth tensor field $J \in \Gamma(\text{End } TM)$ such that $J_p^2 = \epsilon \text{Id}_{T_p M}$ for every point $p \in M$, and in the case where $\epsilon = 1$ the dimensions of the eigendistributions $T^\pm M := \ker(\text{Id} \mp J)$ are required to be equal. We call J almost ϵ -complex structure. One can always find local coordinates at p such that

$$J_p = \begin{pmatrix} 0 & \mathbb{1} \\ \epsilon \mathbb{1} & 0 \end{pmatrix},$$

however this is not necessarily true for any other point in the patch. Almost ϵ -complex manifolds have an even number of dimensions.

An almost ϵ -complex structure induces a dual almost ϵ -complex structure $J^* \in \Gamma(\text{End } T^*M)$, which is defined by

$$\begin{aligned} J^* : T^*M &\longrightarrow T^*M \\ \xi(X) &\longmapsto (J^*\xi)(X) = \xi(JX). \end{aligned}$$

Theorem 2.2 (Newlander–Nirenberg). Consider the complex distribution given by $\mathcal{D} = T^{1,0}M^\mathbb{C}$ on an almost complex manifold ($\epsilon = -1$). The distribution \mathcal{D} is integrable iff

$$[\Gamma(\mathcal{D}), \Gamma(\mathcal{D})] \subset \Gamma(\mathcal{D}).$$

It follows by complex conjugation that $T^{0,1}M^\mathbb{C}$ is also an integrable distribution, and the almost complex structure J itself is said to be integrable.

Remark 2.1. The Newlander–Nirenberg theorem can be reformulated in terms of one-forms as follows: Consider n linearly independent one-forms θ^a such that $\mathcal{D} = \cap^a \ker(\theta^a)$. The distribution \mathcal{D} is integrable iff

$$d\theta^a = C_{bc}^a \theta^b \wedge \varphi^c,$$

where φ^c are arbitrary one-forms.

Theorem 2.3 (Frobenius). Consider the real distributions $\tilde{\mathcal{D}}^+ = T^+M$ and $\tilde{\mathcal{D}}^- = T^-M$ on an almost para-complex manifold ($\epsilon = 1$). The distribution $\tilde{\mathcal{D}}^\pm$ is integrable iff

$$[\Gamma(\tilde{\mathcal{D}}^\pm), \Gamma(\tilde{\mathcal{D}}^\pm)] \subset \Gamma(\tilde{\mathcal{D}}^\pm).$$

If both $\tilde{\mathcal{D}}^+$ and $\tilde{\mathcal{D}}^-$ are integrable then the para-complex structure J is integrable.

The integrability condition of both almost complex and almost para-complex structures is characterised by a vanishing Nijenhuis tensor

$$N(X, Y) = -J^2[X, Y] + J[JX, Y] + J[X, JY] - [JX, JY].$$

Definition 2.12 (ϵ -complex manifold). An almost ϵ -complex manifold (M, J) is an ϵ -complex manifold if the almost ϵ -complex structure J is integrable.

Remark 2.2. A differentiable map between two ϵ -complex manifolds $f : (M, J) \rightarrow (M', J')$ is ϵ -holomorphic if $df \circ J = J' \circ df$. In the case where the image of the map is \mathbb{C}_ϵ this defines an ϵ -holomorphic function as a map $f : M \rightarrow \mathbb{C}_\epsilon$ where $df \circ J = i_\epsilon df$.

Definition 2.13 (ϵ -Hermitian manifold). An ϵ -complex pseudo-Riemannian manifold (M, g, J) is ϵ -Hermitian if the metric is compatible with the ϵ -complex structure

$$g_p(X, Y) = g_p(J_p X, J_p Y), \quad X, Y \in T_p M.$$

The metric g itself is said to be an ϵ -Hermitian metric.

The metric on an ϵ -Hermitian manifold induces a non-degenerate two-form $\omega_p(\cdot, \cdot) := \epsilon g_p(J_p \cdot, \cdot)$, called the fundamental two-form. Any ϵ -complex pseudo Riemannian manifold (M, g, J) defines a ϵ -Hermitian manifold (M, \hat{g}, J) with Hermitian form given by $\hat{g} := g - i\omega$. A (-1) -Hermitian manifold (a Hermitian manifold) has signature $(2p, 2q)$, whereas a $(+1)$ -Hermitian manifold (a para-Hermitian manifold) always has neutral signature $(2n, 2n)$.

Definition 2.14 (ϵ -Kähler). An ϵ -Hermitian manifold (M, g, J) is ϵ -Kähler if the fundamental form is closed $d\omega = 0$. Locally ϵ -Kähler manifolds admit an ϵ -Kähler potential $K(z, \bar{z})$ such that the metric and fundamental form are given by [66]

$$g = \text{Re} \left(\frac{\partial^2 K(z, \bar{z})}{\partial z^i \partial \bar{z}^j} dz^i \otimes d\bar{z}^j \right), \quad \omega = 2i_\epsilon \partial \bar{\partial} K = \frac{\partial^2 K(z, \bar{z})}{\partial z^i \partial \bar{z}^j} dz^i \wedge d\bar{z}^j.$$

Theorem 2.4 . An ϵ -Hermitian manifold (M, g, J) is ϵ -Kähler iff the ϵ -complex structure is parallel with respect to the Levi-Civita connection

$$DJ = 0.$$

For proof see for example [67].

Definition 2.15 (affine special ϵ -Kähler). We define an affine special ϵ -Kähler manifold (M, g, J, ∇) as a pseudo-Riemannian manifold (M, g) equipped with a flat, torsion-free ‘special’ connection ∇ and integrable complex ϵ -structure J such that

- (i) ∇g is completely symmetric.
- (ii) $d\omega = 0$.
- (iii) $\nabla \omega = 0$.

Condition (i) tells us that the manifold is Hessian with respect to the special connection ∇ . From condition (ii) we know that it is ϵ -Kähler. The last condition can be seen as a compatibility condition that ensures the Hessian structure coincides with the Kähler structure. In the literature condition (i) is usually replaced by the condition that ∇J is symmetric. However, given (iii) both conditions can be shown to be equivalent. One may also consider affine special complex manifolds without the requirement of being Kähler [76], however such manifolds will not appear in this thesis.

It can easily be shown that the condition of the special connection ∇ to be torsion-free is actually unnecessary, as it is implied by the other conditions [11]. However, since

the torsion-free condition is integral to the definition of special complex and special real manifolds we include it here for clarity. The special connection ∇ is also not unique, and in fact there is an S^1 family of special connections generated by the complex structure [76]

$$\nabla^{(\alpha)} = e^{\alpha J} \circ \nabla \circ e^{-\alpha J} .$$

On an affine special ϵ -Kähler manifold one can introduce ‘special’ ϵ -holomorphic coordinates X^I , which are characterised by having a flat real part with respect to the special connection

$$\nabla \text{Re}(dX^I) = 0 , \quad \nabla_I = \partial_I .$$

The manifold can always be covered by an atlas of special coordinate charts. Special ϵ -holomorphic coordinates corresponding to different special connections in the S^1 family $\nabla^{(\alpha)}$ are related by phase transformations

$$(\nabla^{(0)}, X^I) \longrightarrow (\nabla^{(\alpha)}, e^{i\alpha} X^I) .$$

Proposition 2.3 . On a special ϵ -holomorphic coordinate patch there exists a local ϵ -holomorphic function $F(X)$ such that the Kähler potential is given by

$$K(X, \bar{X}) := i_\epsilon(X^I \bar{F}_I - F_I \bar{X}^I) , \quad (2.16)$$

where we have introduced the notation $F_I = \partial_I F(X)$. The components of the metric are therefore given by

$$N_{IJ} := \frac{\partial^2 K}{\partial X^I \partial \bar{X}^J} = -i_\epsilon(F_{IJ} - \bar{F}_{IJ}) = -\epsilon 2\text{Im}(F_{IJ}) . \quad (2.17)$$

The function $F(X)$ is known as the ϵ -holomorphic prepotential.

The functions F_I also form an ϵ -holomorphic special coordinate system on the affine special Kähler manifold, i.e. $\nabla \text{Re}(dF_I) = 0$. The two special coordinate systems X^I and F_I are conjugate to one another in the sense that their real parts form a flat Darboux coordinate system, i.e. $\omega = \text{Re}(X^I) \wedge \text{Re}(F_I)$.

Proof. For an affine special Kähler manifold this was proved by Freed in [11], and for an affine special para-Kähler manifold by Cortés et al in [72]. □

Any two special ϵ -holomorphic coordinate systems (X^I, F_I) and $(\tilde{X}^I, \tilde{F}_I)$ are related linearly by a real symplectic transformation and a complex translation

$$\begin{pmatrix} X^I \\ F_I \end{pmatrix} = P \begin{pmatrix} \tilde{X}^I \\ \tilde{F}_I \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} , \quad P \in Sp(2n, \mathbb{R}) , \quad a, b \in \mathbb{C}^n . \quad (2.18)$$

However, the second derivatives of the prepotential transform fractionally linearly

$$F_{IJ} = \left[\left(D\tilde{F} + C \right) \left(B\tilde{F} + A \right)^{-1} \right]_{IJ} ,$$

where we have made the decomposition

$$P = \begin{pmatrix} A & B \\ C & D \end{pmatrix} .$$

Remark 2.3. We define the real coordinates x^I, y_I to be the real parts of X^I, F_I

$$x^I = \operatorname{Re}(X^I), \quad y_I = \operatorname{Re}(F_I).$$

These define a flat Darboux coordinate system, i.e.

$$\omega = dx^I \wedge dy_I, \quad \nabla dx^I = \nabla dy_I = 0.$$

Transition functions are of the form

$$\begin{pmatrix} x^I \\ y_I \end{pmatrix} = P \begin{pmatrix} \tilde{x}^I \\ \tilde{y}_I \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}, \quad P \in Sp(2n, \mathbb{R}), \quad a, b \in \mathbb{R}^n.$$

We will refer to x^I, y_I as special real coordinates. Affine special ϵ -Kähler manifolds are Hessian manifolds, and the Hessian structure is compatible with the special real coordinates. We will come back to this point in section 6.1, where we will investigate the real formulation of affine special ϵ -Kähler geometry in much more detail. For now we will continue with the traditional formulation of affine special ϵ -Kähler geometry in terms of special ϵ -holomorphic coordinates.

Definition 2.16 (conic affine special ϵ -Kähler). A conic affine special ϵ -Kähler manifold (N, g_N, J, ∇, ξ) is an affine special ϵ -Kähler manifold endowed with a vector field ξ that satisfies

- (i) $D\xi = \mathbb{1}$, where D is the Levi-Civita connection.
- (ii) $\nabla\xi = \mathbb{1}$.

Conic affine special Kähler manifolds are clearly a special class of 2-conic Hessian manifolds. They actually admit two distinguished vector fields: ξ and $J\xi$, which commute with one another, and define a distribution $\mathcal{D} = \operatorname{span}\{\xi, J\xi\}$ on the tangent bundle. This has dimension 2 so long as $g_N(\xi, \xi) = g_N(J\xi, J\xi)$ is point-wise non-zero, which we will assume from now on. The Lie derivative of the J with respect to ξ can be written as

$$(\mathcal{L}_\xi J)X = (D_\xi J)X - J(D_X \xi) + D_{JX} \xi,$$

for any smooth vector field $X \in \Gamma(TM)$. Since N is Kähler we have $DJ = 0$, and from condition (i) we have $D_X \xi = X$ and $D_{JX} \xi = JX$. The Lie derivative therefore vanishes

$$\mathcal{L}_\xi J = 0,$$

which implies that ξ is ϵ -holomorphic. By a similar argument one can show that $J\xi$ is also ϵ -holomorphic. In addition, the vector fields ξ and $J\xi$ define an ϵ -holomorphic action of a two-dimensional abelian Lie algebra. ξ generates homotheties whereas $J\xi$ generates isometries

$$\mathcal{L}_\xi g = 2g, \quad \mathcal{L}_{J\xi} g = 0.$$

We will assume that this infinitesimal action lifts to a principal \mathbb{C}_ϵ^* -action on N with base manifold $\bar{N} = N/\mathbb{C}_\epsilon^*$.

We can analyse condition (i) in precisely the same way as for a \mathfrak{d} -conic Hessian manifold, with the particular choice of $\mathfrak{d} = 2$. This means that there always exists real coordinates (r, x^i) such that

$$g_N = dr \otimes dr + r^2 \bar{g}'_{ij}(x) dx^i \otimes dx^j, \quad \xi = r \frac{\partial}{\partial r}.$$

Again, by defining the new coordinates $p^a = (r, rx^i)$ the vector field becomes an Euler field

$$\xi = p^a \frac{\partial}{\partial q_a} ,$$

and the components of the metric in these coordinates are homogeneous functions of degree zero

$$\xi g_{ab}(p) = 0 .$$

Condition (ii) ensures that ξ is the Euler vector field associated with the special real coordinates $q^a = (x^I, y_J)$

$$\xi g_{ab}(q) = 0 .$$

In special holomorphic coordinates X^I the vector field ξ and $J\xi$ take the form

$$\xi = X^I \frac{\partial}{\partial X^I} + \bar{X}^I \frac{\partial}{\partial \bar{X}^I} , \quad J\xi = iX^I \frac{\partial}{\partial X^I} - i\bar{X}^I \frac{\partial}{\partial \bar{X}^I} .$$

The prepotential is a homogeneous function of degree 2, i.e. $F(\lambda X) = \lambda^2 F(X)$ for $\lambda \in \mathbb{C}_\epsilon$.

It is useful to define the rank two tensor field g by

$$g = \frac{\partial^2 \mathcal{K}}{\partial X^I \partial \bar{X}^J} dX^I \otimes d\bar{X}^J ,$$

where $\mathcal{K}(X, \bar{X}) = -\log K(X, \bar{X})$ with K the ϵ -Kähler potential. We can write the components of g as

$$g_{I\bar{J}} = \frac{\partial^2 \mathcal{K}}{\partial X^I \partial \bar{X}^J} = -\frac{N_{IJ}}{\bar{X}NX} + \frac{(N\bar{X})_I (NX)_J}{(\bar{X}NX)^2} . \quad (2.19)$$

One may easily show that this tensor field is degenerate along the subspace \mathcal{D} spanned by ξ and $J\xi$ from the fact that

$$X^I g_{I\bar{J}} = g_{I\bar{J}} \bar{X}^J = 0 . \quad (2.20)$$

Both vector fields ξ and $J\xi$ act as ‘isometries’ of this tensor field

$$\mathcal{L}_\xi g = \mathcal{L}_{J\xi} g = 0 .$$

Definition 2.17 (projective special ϵ -Kähler). A projective special ϵ -Kähler manifold $(\bar{N}, \bar{g}, \bar{J}, \bar{\nabla})$ is defined to be the quotient manifold N/\mathbb{C}_ϵ^* of a conic affine special ϵ -Kähler manifold (N, g, ξ, J, ∇) . The metric \bar{g} on \bar{N} is induced by the tensor field g on N , and the complex structure and connection $\bar{J}, \bar{\nabla}$ on \bar{N} are induced by J, ∇ on N .

One can locally define a projective special ϵ -Kähler manifold as a codimension 2 hypersurface in a conic affine special ϵ -Kähler manifold by imposing any appropriate constraints that fix the homothety ξ and isometry $J\xi$. Such a choice can be given by imposing that the ϵ -Kähler potential K is constant, and the phase of the special coordinates X^I is constant. A particularly useful choice is to select the hypersurface $K = 1$ and $\text{Im}(X^0) = 0$, so we have

$$\bar{M} \simeq \{K = 1, \text{Im}(X^0) = 0\} \subset M .$$

This hypersurface can be parametrised by a set of projective coordinates $z^A = X^A/X^0$, which satisfy the relations

$$z^A = \frac{X^A}{X^0}, \quad |X^0|^2 = i_\epsilon [2\mathcal{F} - 2\bar{\mathcal{F}} - (z^A - \bar{z}^A)(\mathcal{F}_A + \bar{\mathcal{F}}_A)] , \quad \text{Im}(X^0) = 0 ,$$

where $\mathcal{F}_A = \frac{\partial \mathcal{F}}{\partial z^A}$ and

$$\mathcal{F}(z) := F\left(1, \frac{X^1}{X^0}, \dots, \frac{X^n}{X^0}\right) .$$

These are completely analogous to the coordinates ϕ^x introduced in the previous section, which parametrise the hypersurface $H = 1$ on a projective special real manifold. The ϵ -Kähler potential on \bar{M} can be written in terms of projective coordinates as

$$\bar{K} = -\log\left(-i_\epsilon [2\mathcal{F} - 2\bar{\mathcal{F}} - (z^A - \bar{z}^A)(\mathcal{F}_A + \bar{\mathcal{F}}_A)]\right) ,$$

and the metric itself is given by

$$\bar{g} = \text{Re}\left(\frac{\partial^2 \bar{K}(z, \bar{z})}{\partial z^A \partial \bar{z}^B} dz^A \otimes d\bar{z}^B\right) .$$

2.4 (Para-)quaternion Kähler manifolds

In this section we will discuss quaternion Kähler and para-quaternion Kähler manifolds. Quaternion Kähler manifolds appear as the target manifold of $4d$, $\mathcal{N} = 2$ hyper multiplets coupled to supergravity [77]. It is expected that para-quaternion Kähler manifolds appear as the target manifold of the Euclidean version of such theories, as this is the case for certain symmetric space examples obtained by dimensional reduction over time [78, 51]. We will later prove that all Euclidean hyper multiplets obtained from dimensional reduction over time are para-quaternion Kähler, which is one of the main results of this thesis. Some standard references on quaternion Kähler manifolds are [79, 66, 80]. For para-quaternion Kähler manifolds we follow the definition given in [81].

In the previous section we discussed both Kähler and para-Kähler manifolds in a unified way using ϵ -notation. One can also take this approach for quaternion and para-quaternion Kähler manifolds, e.g. [81], as the two types of manifold do indeed share many similar features. However, for applications in this thesis they exhibit enough differences to warrant a separate discussion for each case, and this is the approach we will take here.

Before we give the definition of a quaternion Kähler manifold we first define the pseudo-unitary-symplectic group $\text{Sp}(k, l)$ by

$$\text{Sp}(k, l) = \text{U}(2k, 2l) \cap \text{Sp}(2k + 2l, \mathbb{C}) \subset \text{SO}(4k, 4l) .$$

It follows that $\text{Sp}(1) = \text{U}(2) \cap \text{Sp}(2, \mathbb{C}) = \text{SU}(2)$. We define the product of these groups by

$$\text{Sp}(k, l) \cdot \text{Sp}(1) = (\text{Sp}(k, l) \times \text{Sp}(1))/\mathbb{Z}_2 ,$$

where the \mathbb{Z}_2 corresponds to $\pm \text{Id}$. The group $\text{Sp}(k, l) \cdot \text{Sp}(1)$ acts on a set of m quaternions $x^\mu \in \mathbb{H}^m$ through left multiplication by $A \in \text{Sp}(k, l)$ and right multiplication by $\lambda \in \text{Sp}(1)$, i.e.

$$\begin{aligned} \text{Sp}(k, l) \cdot \text{Sp}(1) \times \mathbb{H}^m &\longrightarrow \mathbb{H}^m \\ (A^\mu{}_\nu, \lambda; x^\mu) &\longmapsto A^\mu{}_\nu x^\nu \lambda^\dagger . \end{aligned}$$

Definition 2.18 (quaternion Kähler). A $4n$ -dimensional pseudo-Riemannian manifold (M, g) is a quaternion Kähler manifold if the Riemannian holonomy group $\text{Hol}(D)$ is a subgroup of $\text{Sp}(k, l) \cdot \text{Sp}(1)$.

On a quaternion Kähler manifold there exists a quaternion structure given by $Q = \text{span}\{J_1, J_2, J_3\} \subset \text{End}(TM)$, which is generated by three metric compatible almost complex structures J_1, J_2 and J_3 that satisfy the quaternion relations

$$J_1^2 = J_2^2 = J_3^2 = -1, \quad J_1 J_2 = J_3 = -J_2 J_1.$$

The Levi-Civita connection preserves this structure in the sense that the covariant derivative of J_α is a linear combination of J_1, J_2, J_3 , which from the quaternion relations implies that

$$\begin{aligned} D_X J_1 &= \alpha(X) J_2 + \beta(X) J_3 \\ D_X J_2 &= -\alpha(X) J_1 + \gamma(X) J_3 \\ D_X J_3 &= -\beta(X) J_1 - \gamma(X) J_2 \end{aligned}$$

where $\alpha, \beta, \gamma \in \Gamma(T^*M)$. At a point in the overlap of charts $p \in U_i \cap U_j$ the quaternion structure $Q_p \subset \text{End}(T_p M)$ on U_i and U_j agree with one another, however the almost complex structures themselves need not be the same, i.e. $J_1|_{U_i} \neq J_1|_{U_j}$ etc.

A quaternion Kähler manifold necessarily has $4n$ real dimensions and metric signature $(4k, 4l)$. Since for the particular case $n = 1$ the restriction $\text{Hol}(D) \subset \text{Sp}(1) \cdot \text{Sp}(1)$ only implies that the manifold is oriented, the following stronger definition is used instead: a 4-dimensional pseudo-Riemannian manifold is quaternion Kähler if it is an oriented Einstein manifold with self-dual Weyl tensor. This ensures that for any quaternion Kähler manifold we have

$$R_{\mu\nu} = R(n+2)g_{\mu\nu},$$

and since the scalar curvature R of a quaternion Kähler manifold is always constant all quaternion Kähler manifolds are Einstein manifolds [79]. Despite their name, quaternion Kähler manifolds are in general neither Kähler manifolds nor even complex manifolds.

A practical way to show that a manifold is quaternion Kähler is to compute the Levi-Civita connection one-form. One must make the decomposition

$$\omega = p \otimes \mathbb{1}_{(2k, 2l)} + \mathbb{1}_2 \otimes \begin{pmatrix} q & t \\ -\bar{t} & \bar{q} \end{pmatrix},$$

where the components of p, q, t are one-forms, and check that

$$\begin{aligned} \begin{pmatrix} q & t \\ -\bar{t} & \bar{q} \end{pmatrix} \in \mathfrak{sp}(k, l) & \Leftrightarrow \begin{aligned} q^\dagger \begin{pmatrix} -\mathbb{1}_k & 0 \\ 0 & \mathbb{1}_l \end{pmatrix} &= - \begin{pmatrix} -\mathbb{1}_k & 0 \\ 0 & \mathbb{1}_l \end{pmatrix} q \\ t^T \begin{pmatrix} -\mathbb{1}_k & 0 \\ 0 & \mathbb{1}_l \end{pmatrix} &= - \begin{pmatrix} -\mathbb{1}_k & 0 \\ 0 & \mathbb{1}_l \end{pmatrix} t \end{aligned} \end{aligned}$$

and

$$p \in \mathfrak{sp}(1) \Leftrightarrow \text{Tr}(p) = 0, \quad p^\dagger = -p.$$

If these conditions are satisfied then $\omega \in (\mathfrak{sp}(p, q) \oplus \mathfrak{sp}(1)) \otimes T^*M$ and by corollary 2.1 the manifold is quaternion Kähler.

Definition 2.19 (para-quaternion Kähler). A $4n$ -dimensional pseudo-Riemannian manifold (M, g) is a para-quaternion Kähler manifold if the Riemannian holonomy group $\text{Hol}(D)$ is a subgroup of $\text{Sp}(2k + 2l, \mathbb{R}) \cdot \text{Sp}(2, \mathbb{R})$.

On a para-quaternion Kähler manifold there exists a para-quaternion structure $\tilde{Q} = \text{span}\{\tilde{J}_1, \tilde{J}_2, \tilde{J}_3\} \subset \text{End}(TM)$ that is generated by two metric compatible almost para-complex structures \tilde{J}_1, \tilde{J}_2 and one metric compatible almost complex structure \tilde{J}_3 that satisfy the para-quaternion (or ‘split’ quaternion) relations

$$\tilde{J}_1^2 = \tilde{J}_2^2 = 1, \quad \tilde{J}_3^2 = -1, \quad \tilde{J}_1\tilde{J}_2 = \tilde{J}_3 = -\tilde{J}_2\tilde{J}_1.$$

The Levi-Civita connection preserves the para-quaternion structure in the sense that the covariant derivative of \tilde{J}_α is a linear combination of $\tilde{J}_1, \tilde{J}_2, \tilde{J}_3$, which from the para-quaternion relations implies that

$$\begin{aligned} D_X \tilde{J}_1 &= \alpha(X)\tilde{J}_2 + \beta(X)\tilde{J}_3 \\ D_X \tilde{J}_2 &= -\alpha(X)\tilde{J}_1 + \gamma(X)\tilde{J}_3 \\ D_X \tilde{J}_3 &= \beta(X)\tilde{J}_1 + \gamma(X)\tilde{J}_2 \end{aligned}$$

where $\alpha, \beta, \gamma \in \Gamma(T^*M)$. At a point in the overlap of charts $p \in U_i \cap U_j$ the para-quaternion structure $\tilde{Q}_p \subset \text{End}(T_p M)$ on U_i and U_j agree with one another, however the almost (para-)complex structures themselves need not be the same, i.e. $\tilde{J}_1|_{U_i} \neq \tilde{J}_1|_{U_j}$ etc.

As for a quaternion Kähler manifold, a para-quaternion Kähler manifold necessarily has $4n$ real dimensions and neutral metric signature $(2n, 2n)$. Since for $n = 1$ the restriction $\text{Hol}(D) \subset \text{Sp}(2, \mathbb{R}) \cdot \text{Sp}(2, \mathbb{R})$ only implies that the manifold is oriented we use the stronger definition that a 4-dimensional pseudo-Riemannian manifold is para-quaternion Kähler if it is an oriented Einstein manifold with self-dual Weyl tensor. This ensures that for any para-quaternion Kähler manifold we have

$$R_{\mu\nu} = R(n + 2)g_{\mu\nu},$$

and since the scalar curvature R of a para-quaternion Kähler manifold is always constant all para-quaternion Kähler manifolds are Einstein manifolds. Para-quaternion Kähler manifolds are in general neither Kähler manifolds nor even para-complex manifolds.

A practical way to show that a manifold is para-quaternion Kähler is to compute the Levi-Civita connection one-form. One must make the decomposition

$$\omega = p \otimes \mathbb{1}_{(2k, 2l)} + \mathbb{1}_2 \otimes \begin{pmatrix} q & t \\ -t^T & -q^T \end{pmatrix},$$

where the components of p, q, t are one-forms, and check that

$$\begin{pmatrix} q & t \\ -t^T & -q^T \end{pmatrix} \in \mathfrak{sp}(2k + 2l, \mathbb{R}) \quad \Leftrightarrow \quad t^T \begin{pmatrix} -\mathbb{1}_k & 0 \\ 0 & \mathbb{1}_l \end{pmatrix} = \begin{pmatrix} -\mathbb{1}_k & 0 \\ 0 & \mathbb{1}_l \end{pmatrix} t,$$

and

$$p \in \mathfrak{sp}(2, \mathbb{R}) \quad \Leftrightarrow \quad \text{Tr}(p) = 0.$$

If these conditions are satisfied then $\omega \in (\mathfrak{sp}(2k + 2l, \mathbb{R}) \oplus \mathfrak{sp}(2, \mathbb{R})) \otimes T^*M$ and by corollary 2.1 the manifold is para-quaternion Kähler.

Chapter 3

Preliminary physics

We will now introduce the necessary background physics needed in order to understand the main part of this thesis. This includes an introduction to $\mathcal{N} = 2$ supergravity, both from the point of view of the supersymmetry algebra and the Lagrangian, and a primer on black hole physics. We must also introduce the reader to the procedure of dimensional reduction. We will take some time to go through the calculations necessary for the r-map and c-map fully and explicitly, as they are often only summarised in the literature.

We discuss $\mathcal{N} = 2$ supergravity in section 3.1, and present the necessary Lagrangians we will need to use later on in the thesis. In section 3.2 we discuss three important black hole solutions: the Schwarzschild, Reissner–Nordström and Kerr–Newman solutions. We then discuss how these can be embedded into theories of extended supergravity. We lastly discuss dimensional reduction in section 3.3, summarising the most important results at the beginning of this section.

3.1 $\mathcal{N} = 2$ supergravity

In this thesis we will deal exclusively with theories of $\mathcal{N} = 2$ supergravity (along with a particular generalisation thereof in five dimensions). Here we will provide an introduction to the subject. This discussion is designed to be self-contained, and should be understandable even if the previous chapter has been skipped. However, we shall try to link in with some of the geometrical concepts introduced in the previous chapter when it provides additional illumination on the subject at hand.

In section 3.1.1 we will introduce the supersymmetry algebra, and discuss massive and massless representations. We then present the Lagrangians of five-dimensional vector multiplets, four-dimensional vector multiplets and hyper multiplets in sections 3.1.2, 3.1.3 and 3.1.4 respectively.

3.1.1 Algebra and representations

We will focus on the supersymmetry algebra and representations in four spacetime dimensions, as it is the most physically relevant. Although the five-dimensional algebra and representations will also be used in this thesis, the discussion is close enough to the four-dimensional case that it will not be repeated here. We refer the reader to the many comprehensive texts on supersymmetry in five-dimensions and higher, e.g. [82, 83, 14].

It is understood that the laws of physics should be invariant under Poincaré transformations. These transformations are generated by the operators P_μ (translations) and

$M_{\mu\nu}$ (Lorentz transformations), which together form the Lie algebra of the Poincaré group

$$\begin{aligned} [P_\mu, P_\nu] &= 0, \\ [M_{\mu\nu}, P_\rho] &= i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu), \\ [M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}). \end{aligned}$$

A famous no-go theorem by Coleman and Mandula states that any symmetry of the S-matrix with bosonic generators must commute with the Lie algebra of the Poincaré group, and so any other non-trivial symmetry must be internal (i.e. the generators transform as scalars) [84, 85]. However, one can get round this theorem by introducing fermionic generators that anti-commute with one another but do not commute with all elements of the Poincaré algebra. A theorem by Haag, Lopuszanski and Sohnius [86] tells us that the only way to do this consistently is to extend the Poincaré algebra into a so-called super Poincaré algebra by introducing fermionic generators Q_α^A , which satisfy

$$\begin{aligned} [P_\mu, Q_\alpha^A] &= 0, \\ \{Q_\alpha^A, \bar{Q}_\beta^B\} &= 2\delta^{AB}(\sigma_\mu)_{\alpha\dot{\beta}}P^\mu, \\ \{Q_\alpha^A, Q_\beta^B\} &= \epsilon_{\alpha\beta}Z^{AB}, \end{aligned}$$

where $\bar{Q}_\alpha^A = \epsilon_{\dot{\alpha}}^\beta (Q_\beta^A)^*$ and spinorial indices are raised and lowered with $\delta_{\alpha\dot{\beta}}$. The matrices σ_μ are given by

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The generators $Q_\alpha^A, \bar{Q}_\alpha^A$ transform as two-component Weyl spinors under Lorentz transformations, with chirality $+1/2, -1/2$ respectively. This completely fixes their commutation relations with the generators of Lorentz transformations

$$\begin{aligned} [M_{\mu\nu}, Q_\alpha^A] &= -i\frac{1}{2}(\sigma_{\mu\nu})_\alpha^\beta Q_\beta^A, \\ [M_{\mu\nu}, \bar{Q}_\alpha^A] &= -i\frac{1}{2}(\bar{\sigma}_{\mu\nu})_{\dot{\alpha}}^{\dot{\beta}} \bar{Q}_{\dot{\beta}}^A, \end{aligned}$$

where $\sigma_{\mu\nu} = \frac{1}{2}\sigma_{[\mu}\sigma_{\nu]}^1$. The operators Z^{AB} commute with all elements of the super Poincaré algebra and are therefore called central charges. The indices A, B run from $1, \dots, \mathcal{N}$, where \mathcal{N} measures the extent to which the theory is supersymmetric, or, more precisely, there are $4\mathcal{N}$ real supercharges associated with the theory [87]. When the central charges vanish the supersymmetry algebra is invariant under $U(\mathcal{N})$ ‘R-symmetry’ automorphisms $Q_\alpha^A \rightarrow S^A_B Q_\alpha^B$.

A theory that respects the symmetries generated by the super Poincaré algebra and has a non-dynamical spacetime metric is called a theory of rigid (or global) supersymmetry. If we include gravity, i.e. allow the spacetime metric to become a dynamical field, we call it instead a theory of supergravity (or local supersymmetry).

The symmetries generated by $Q_\alpha^A, \bar{Q}_\alpha^A$ transform particles of different spin into one another. Since the supersymmetry generators commute with the momentum operator P_μ , in all irreducible representations of the super Poincaré algebra any bosonic (rsp.

¹Note that $J_i = \{M_{23}, M_{31}, M_{12}\}$, $\sigma_i = -i\frac{1}{2}\{\sigma_{23}, \sigma_{31}, \sigma_{12}\}$ and so we have $[J_i, Q_\alpha^A] = -\frac{1}{2}(\sigma_i)_\alpha^\beta Q_\beta^A$.

fermionic) particle is always accompanied by at least one fermionic (rsp. bosonic) particle of the same mass. A closed set of bosonic and fermionic particles that transform into one another under supersymmetry transformations is called a multiplet, and the particular types of multiplet allowed in a theory depends on the value of \mathcal{N} .

In this thesis we are primarily concerned with theories that have $\mathcal{N} = 2$ supersymmetry, which we will now exclusively focus on. In this case the super Poincaré algebra reduces to

$$\begin{aligned}\{Q_\alpha^A, \bar{Q}_{\dot{\beta}}^B\} &= 2\delta^{AB}(\sigma_\mu)_{\alpha\dot{\beta}}P^\mu, \\ \{Q_\alpha^1, Q_\beta^2\} &= -\{Q_\alpha^2, Q_\beta^1\} = 2\epsilon_{\alpha\beta}|Z|.\end{aligned}$$

Here we have defined $2|Z| := |Z^{12}|$, and we have made a U(1) phase transformation of the supersymmetry generators to eliminate the phase of the central charge Z^{12} .

Massive representations

For massive representations we choose a frame in which $P_\mu = (-M, 0, 0, 0)$, and so

$$\sigma^\mu P_\mu = M\sigma_0 = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}. \quad (3.1)$$

In this case we can write the algebra as

$$\begin{aligned}\{Q_\alpha^A, \bar{Q}_\beta^B\} &= 2M\delta_{\alpha\beta}\delta^{AB}, \\ \{Q_\alpha^1, Q_\beta^2\} &= -\{Q_\alpha^2, Q_\beta^1\} = 2\epsilon_{\alpha\beta}|Z|.\end{aligned}$$

It is useful to define the following linear combination of supersymmetry generators [87]

$$a_\alpha = \frac{1}{\sqrt{2}} \left(Q_\alpha^1 + \epsilon_\alpha^{\dot{\beta}} \bar{Q}_{\dot{\beta}}^2 \right), \quad b_\alpha = \frac{1}{\sqrt{2}} \left(Q_\alpha^1 - \epsilon_\alpha^{\dot{\beta}} \bar{Q}_{\dot{\beta}}^2 \right),$$

which satisfy

$$\begin{aligned}\{a_\alpha, \bar{a}_{\dot{\beta}}\} &= 2(M + |Z|)\delta_{\alpha\dot{\beta}}, \\ \{b_\alpha, \bar{b}_{\dot{\beta}}\} &= 2(M - |Z|)\delta_{\alpha\dot{\beta}}.\end{aligned}$$

Irreducible massive representations of the Poincaré algebra are identified by the spin s of a particle (i.e. its representation under the little group $\text{SO}(3)$), upon which a_α, b_α act as annihilation operators and $\bar{a}_{\dot{\alpha}}, \bar{b}_{\dot{\alpha}}$ act as creation operators. A basis of irreducible representations is therefore given by

$$\bar{a}_{\dot{\alpha}_1} \dots \bar{b}_{\dot{\beta}_1} \dots |s\rangle,$$

where

$$a_\alpha |s\rangle = b_\alpha |s\rangle = 0.$$

In order to avoid negative norm states we need $M \geq |Z|$. States which saturate this bound are known as Bogomol'nyi–Prasad–Sommerfield (BPS) states.

For massive representations in which $M > |Z|$ the full set of supersymmetry generators act non-trivially, and since we have 4 creation operators we have $2^4 = 16$ states in a representation. This is not true if the BPS bound is satisfied, i.e. $M = |Z|$. In this case representations can be chosen such that the $b_\alpha, \bar{b}_{\dot{\alpha}}$ annihilate all states, and therefore such states remain invariant under half the supersymmetry transformations. Since we have 2 creation operators we only have $2^2 = 4$ states in these so-called ‘BPS’ multiplets.

Massless representations

Massless states are classified in terms of helicity, which characterises their representation in terms of the little group $SO(2)$. We can obtain states with different helicity by acting on the state with the supersymmetry generators $Q_\alpha^A, \bar{Q}_{\dot{\alpha}}^A$. Since these operators commute with P_μ they have the same four-momentum, and we can always choose a frame in which $P_\mu = (-E, 0, 0, E)$, i.e.

$$\sigma^\mu P_\mu = E(\sigma_0 + \sigma_3) = \begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix}. \quad (3.2)$$

This implies that $\{Q_{-1/2}^A, \bar{Q}_{-1/2}^B\} = 0$, and therefore $Q_{-1/2}^A$ and $\bar{Q}_{-1/2}^B$ and the central charge Z annihilate all states. The remaining supersymmetry generators $Q_{1/2}^A$ and $\bar{Q}_{1/2}^A$ lower and raise helicity by $1/2$ respectively, and so we have $2^2 = 4$ states in each representation. Allowing the lowering operators² $\frac{1}{2\sqrt{E}}Q_{1/2}^1, \frac{1}{2\sqrt{E}}Q_{1/2}^2$ to act on a state of maximum helicity gives four distinct states with the same four-momentum:

helicity	state	
λ_{\max}	$ \lambda_{\max}\rangle$	
$\lambda_{\max} - 1/2$	$\frac{1}{2\sqrt{E}}Q_{1/2}^1 \lambda_{\max}\rangle$	$\frac{1}{2\sqrt{E}}Q_{1/2}^2 \lambda_{\max}\rangle$
$\lambda_{\max} - 1$	$\frac{1}{4E}Q_{1/2}^1Q_{1/2}^2 \lambda_{\max}\rangle$	

These four states form a basis of an irreducible representation of the supersymmetry algebra. Note that since the central charge annihilates every state the operators $Q_{1/2}^1, Q_{1/2}^2$ anti-commute. The possible base states of massless representations of the $\mathcal{N} = 2$ super-Poincaré algebra contain the following combinations of helicities [87]

		λ_{\max}						
		2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1
helicity	2	1						
	$\frac{3}{2}$	2	1					
	1	1	2	1				
	$\frac{1}{2}$		1	2	1			
	0			1	2	1		
	$-\frac{1}{2}$				1	2	1	
	-1					1	2	1
	$-\frac{3}{2}$						1	2
	-2							1

²The factors of $2\sqrt{E}$ have been included to ensure the correct normalisation.

A state with helicity $+1$ belongs to a massless vector boson, which transforms into a state of helicity -1 under CP transformations. Therefore a physical representation must include states with helicities $+1$ and -1 . A massless vector multiplet is therefore obtained by combining the representation with maximum helicity $+1$ with a representation of maximum helicity 0 . It contains a massless vector field A_μ , two Weyl spinors λ^1, λ^2 and a complex scalar field ϕ . By the same argument the gravity multiplet is obtained through the representations with maximum helicities 2 and -1 , and consists of the graviton $g_{\mu\nu}$, two Weyl spinors ψ_μ^1, ψ_μ^2 called gravitini, and the graviphoton gauge field A_μ . A hyper multiplet consists two representations with maximum helicity $1/2$, and contains two Weyl spinors γ^1, γ^2 and four real scalar fields q^1, q^2, q^3, q^4 .

Our discussion now switches focus from the $\mathcal{N} = 2$ supersymmetry algebra and representations to the Lagrangian and field content of $\mathcal{N} = 2$ supergravity coupled to vector and hyper multiplets. The scalar fields in these Lagrangian are hugely important to our discussion, as they form of so-called non-linear sigma models (see appendix A) into a specific type of target manifold associated with these theories. These target manifolds are given precisely by the special geometries introduced in the previous chapter, which is where the name comes from.

The Lagrangian of $\mathcal{N} = 2$ supergravity coupled to vector multiplets takes a slightly different form in five and four dimensions, and so must be discussed separately. For five-dimensional vector multiplets the corresponding target manifold is an affine special real or projective special real manifold, depending on whether the theory has rigid or local supersymmetry respectively. For four-dimensional vector multiplets this is an affine special Kähler or projective special Kähler manifold. For hyper multiplets in three, four or five dimensions the target manifold is always a hyper Kähler manifold for rigid supersymmetry or a quaternion Kähler manifold for local supersymmetry.

In this thesis we are concerned exclusively in theories with local supersymmetry, so we will only consider vector and hyper multiplets coupled to supergravity. We will begin by discussing the Lagrangian of five-dimensional vector multiplets coupled to supergravity, before moving on to vector multiplets. We end with the Lagrangian for hyper multiplets coupled to supergravity in either three, four or five dimensions, for which the bosonic part is always the same.

3.1.2 Five-dimensional vector multiplets

A vector multiplet in five dimensions is built out of a vector A_μ , a real scalar ϕ , and two spinors. The gravity multiplet consists of the five-dimensional metric $g_{\mu\nu}$, vector field A_μ , and two gravitini. We will consider n vector multiplets coupled to the supergravity multiplet, so in total the bosonic Lagrangian consists of n scalar fields ϕ^x , $(n+1)$ vector fields A_μ^I , and the metric. It is given by [69, 88, 27, 14]

$$\begin{aligned} \mathbf{e}^{-1} \mathcal{L}_5 = & \frac{1}{2} R - \frac{3}{4} \bar{a}_{xy}(\phi) \partial_\mu \phi^x \partial^\mu \phi^y - \frac{1}{4} a_{IJ}(\phi) F_{\mu\nu}^I F^{J\mu\nu} \\ & + \frac{1}{6\sqrt{6}} C_{IJK} F_{\mu\nu}^I F_{\rho\sigma}^J A_\lambda^K \mathbf{e}^{-1} \epsilon^{\mu\nu\rho\sigma\lambda} . \end{aligned} \quad (3.3)$$

The constant coefficients C_{IJK} completely determine the dynamics of the Lagrangian. The coupling matrices are defined as follows: first we define the $(n+1)$ scalar fields h^I by

$$h^x = \phi^x h^0 , \quad h^0 = H (1, \phi^1, \dots, \phi^n)^{-\frac{1}{3}} , \quad (3.4)$$

where H is completely determined by C_{IJK} through

$$H(h^0, \dots, h^n) = \frac{1}{6} C_{IJK} h^I h^J h^K .$$

The fields ϕ^x parametrise the hypersurface $H(h) = 1$. The couplings of the vector fields are defined as

$$a_{IJ}(h) = -\frac{1}{3} \frac{\partial^2}{\partial h^I \partial h^J} \log H(h^1, \dots, h^n) = -\frac{1}{3} \left(\frac{H_{IJ}}{H} - \frac{H_I H_J}{H^2} \right) ,$$

where $H_I = \partial_I H$, $H_{IJ} = \partial_{I,J}^2 H$. This can be written explicitly in terms of ϕ^x through (3.4). The couplings of the scalar fields are given by

$$\bar{a}_{xy} = a_{IJ}(\phi) \frac{\partial h^I}{\partial \phi^x} \frac{\partial h^J}{\partial \phi^y} .$$

The scalar kinetic term takes the form of a non-linear sigma model, which allows us to give a mathematical interpretation of the couplings that appear in the Lagrangian. The scalar fields can be interpreted as differentiable maps from five-dimensional space-time into an n -dimensional projective special real target manifold, as described in the previous chapter. The positive definite metric a_{xy} on the projective special real manifold is given by the push-forward of the positive definite metric $a_{IJ} = -\frac{1}{3} \partial_{I,J}^2 \log H$ on the associated 3-conic Hessian manifold with Hesse potential H , which is a cubic polynomial with coefficients $\frac{1}{6} C_{IJK}$. In the notation of the previous section we would write

$$\bar{g} = a_{xy} d\phi^x \otimes d\phi^y = \pi_* a , \quad a = a_{IJ} dh^I \otimes dh^J .$$

We can understand the metric H_{IJ} , which has signature $(+ - \dots -)$, as the metric that appears in the definition of a 3-conic Hessian manifold.

The most straight-forward method of deriving the Lagrangian (3.3) is to first derive the Lagrangian of the corresponding superconformal theory, which is somewhat easier, and then to use the fact that it is gauge equivalent to the Lagrangian of the Poincaré theory. Unfortunately this process is rather long and arduous, and to present it in this thesis would require us to deviate too far from the narrative. A comprehensive review of this method can be found in [88], and the reader may also refer to [8, 14] for reviews of the analogous method in four dimensions. We end our discussion of five-dimensional vector multiplets by remarking that one of the gauge fixing conditions, known as the D-gauge, places a hypersurface constraint on the scalar fields appearing in the superconformal theory. We can now understand why we projected the $(n+1)$ scalar fields h^I onto the hypersurface $H(h) = 1$, which we then parametrised by the n physical scalar fields ϕ^x .

3.1.3 Four-dimensional vector multiplets

A vector multiplet in four dimensions with Minkowski (rsp. Euclidean) spacetime signature is built out of one vector field A_μ , one complex (rsp. para-complex) scalar field z , and two spinors. The gravity multiplet is built from the four-dimensional metric $g_{\mu\nu}$, a vector field A_μ , and two gravitini. We will consider n vector multiplets coupled to supergravity, which has bosonic field content consisting of n (para-)complex scalar

fields z^A , $(n+1)$ vector fields A_μ^I and the metric. The bosonic part of the Lagrangian is given by [89, 27, 14]

$$\begin{aligned} \mathbf{e}^{-1}\mathcal{L}_4 = & \frac{1}{2}R - \bar{g}_{AB}(z, \bar{z})\partial_\mu z^A\partial^\mu \bar{z}^B \\ & + \frac{1}{4}\mathcal{I}_{IJ}(z, \bar{z})F_{\mu\nu}^IF^{J\mu\nu} + \frac{1}{4}\mathcal{R}_{IJ}(z, \bar{z})F_{\mu\nu}^I\tilde{F}^{J\mu\nu}. \end{aligned} \quad (3.5)$$

The dynamics of the Lagrangian are completely determined by a (para-)holomorphic function F , called the prepotential, which is homogeneous of degree 2. The coupling matrices are defined as follows: first define the $(n+1)$ (para-)complex scalar fields X^I by

$$\begin{aligned} z^A = \frac{X^A}{X^0}, \quad \bar{z}^A = \frac{\bar{X}^A}{\bar{X}^0}, \quad \text{Im}(X^0) = 0, \\ |X^0| = \left(-i_\epsilon [2\mathcal{F} - 2\bar{\mathcal{F}} - (z^A - \bar{z}^A)(\mathcal{F}_A + \bar{\mathcal{F}}_A)] \right)^{-\frac{1}{2}}, \end{aligned} \quad (3.6)$$

where $\mathcal{F}_A = \frac{\partial}{\partial z^A}\mathcal{F}$ and \mathcal{F} is completely determined by the prepotential $F(X) = F(X^0, \dots, X^n)$ through

$$\mathcal{F}(z) = F(1, z^1, \dots, z^n).$$

The fields z^A parametrise the codimension 2 hypersurface $e^{-\mathcal{K}} = 1, \text{Im}(X^0) = 0$, where

$$e^{-\mathcal{K}} = i_\epsilon (X^I\bar{F}_I - F_I\bar{X}^I) = -N_{IJ}X^I\bar{X}^J, \quad N_{IJ} = -i_\epsilon(F_{IJ} - \bar{F}_{IJ}).$$

The couplings of the gauge fields are defined by

$$\mathcal{N}_{IJ}(X, \bar{X}) = \mathcal{R}_{IJ}(X, \bar{X}) + i_\epsilon\mathcal{I}_{IJ}(X, \bar{X}) = \bar{F}_{IJ}(\bar{X}) - i_\epsilon\frac{(NX)_I(NX)_J}{XNX}, \quad (3.7)$$

where $(NX)_I = N_{IJ}X^J$ and $XNX = N_{IJ}X^IX^J$. This can be written explicitly in terms of z^A, \bar{z}^A using (3.6). In order to define the scalar couplings we must first define $g_{I\bar{J}}$ precisely as in (2.19), which we repeat here:

$$g_{I\bar{J}} = \frac{\partial^2\mathcal{K}}{\partial X^I\partial\bar{X}^J} = -\frac{N_{IJ}}{\bar{X}NX} + \frac{(N\bar{X})_I(NX)_J}{\bar{X}NX^2}. \quad (3.8)$$

This rank two tensor field is degenerate along the directions $\xi = X^I\partial_I + \bar{X}^I\partial_{\bar{I}}$ and $J\xi = \bar{X}^I\partial_{\bar{I}} - \bar{X}^I\partial_{\bar{I}}$ due to (2.20). The couplings for the scalar fields are then defined as

$$\bar{g}_{A\bar{B}} = g_{I\bar{J}}\frac{\partial X^I}{\partial z^A}\frac{\partial\bar{X}^J}{\partial\bar{z}^B} = \frac{\partial^2\bar{K}}{\partial z^A\partial\bar{z}^B},$$

where

$$\bar{K} = -\log \left(-i_\epsilon [2\mathcal{F} - 2\bar{\mathcal{F}} - (z^A - \bar{z}^A)(\mathcal{F}_A + \bar{\mathcal{F}}_A)] \right).$$

As for five-dimensional vector multiplets, the scalar kinetic term is given by a non-linear sigma model. The scalar fields form differentiable maps from four-dimensional spacetime into an $2n$ -dimensional projective special (para-)Kähler target manifold, as defined in the previous chapter. The matrix $\bar{g}_{A\bar{B}}$ is required to be positive definite, and is given by the push-forward of the degenerate tensor field $g_{I\bar{J}}$ on the corresponding conic affine special (para-)Kähler manifold with prepotential F . We can write this as

$$\bar{g} = \text{Re}(\bar{g}_{A\bar{B}}dz^A \otimes d\bar{z}^B) = \pi_*g, \quad g = \text{Re}(g_{I\bar{J}}dX^I \otimes d\bar{X}^J).$$

The matrix \mathcal{I}_{IJ} can be understood as a negative definite metric on the conic affine special (para-)Kähler manifold.

The Lagrangian (3.5) is most clearly derived by gauge fixing the corresponding superconformal theory, as for the five-dimensional case. As mentioned, the reader may refer to [8, 14] for comprehensive and modern reviews of this method. One of the gauge-fixing conditions, known as the D-gauge, places one hypersurface constraint on the scalar fields X^I appearing in the super conformal theory. This corresponds to the hypersurface $e^{-\mathcal{K}} = 1$. In contrast to the five-dimensional theory, we must also fix a $U(1)$ symmetry which corresponds to an overall phase transformation of the scalar fields X^I . We can choose this to be for example $\text{Im}(X^0) = 0$.

3.1.4 Hyper multiplets

Hyper multiplets are built out of 4 real scalar fields q^1, q^2, q^3, q^4 and two spin 1/2 fields. We will consider n hyper multiplets coupled to supergravity. The bosonic part of the Lagrangian consists of $4n$ scalar fields q^u , the graviphoton and the metric, and takes the same form in three, four and five dimensions. The bosonic part of the action decouples into two distinct parts $\mathcal{L} = \mathcal{L}^{\text{EM}} + \mathcal{L}^{\text{scal}}$. This first part is simply the bosonic part of the pure supergravity Lagrangian, which is given by a standard Einstein–Maxwell Lagrangian

$$e^{-1}\mathcal{L}^{\text{EM}} = \frac{1}{2}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

Note that in the three-dimensional case the gauge field can be dualised into a scalar field. The second part consists only of the scalar fields

$$e^{-1}\mathcal{L}^{\text{scal}} = -h_{uv}\partial_\mu q^u\partial^\mu q^v. \quad (3.9)$$

We can consider more generally n hyper multiplets along with n' vector multiplets coupled to supergravity. In this case we would simply replace the \mathcal{L}^{EM} part of the Lagrangian with the Lagrangian of n' vector multiplets coupled to supergravity, as previously discussed.

The couplings of scalar fields can be rather more general than in the previous cases of vector multiplets. For hyper multiplets the matrix h_{uv} must describe a metric on a quaternion Kähler manifold with scalar curvature [77, 14]

$$R = -4n(n+2). \quad (3.10)$$

Recall from the definition of a quaternion Kähler manifold that this means the holonomy group of the Levi-Civita connection is a subgroup of $\text{Sp}(n) \cdot \text{Sp}(1)$.

Clearly we can understand the kinetic term for the scalar fields appearing in the hyper multiplet Lagrangian as a non-linear sigma model from three-, four-, or five-dimensional spacetime into a $4n$ -dimensional quaternion Kähler target manifold, as described in the previous chapter, with scalar curvature given by (3.10). Like in the case for the vector multiplets, the Lagrangian can be derived from super conformal methods [14]. In this case the physical scalar fields q^u of the Poincaré theory parametrise a codimension 4 hypersurface of a hyper Kähler manifold.

3.2 Black holes

In this section we will introduce the three fundamental solutions of Einstein–Maxwell theory: the Schwarzschild, Reissner–Nordström and Kerr–Newman black holes. This

not only gives us the opportunity to discuss various interesting aspects of black hole physics, but also all new solutions that we present will be direct generalisations of these black holes. Supergravity theories necessarily introduce more fields into the Lagrangian, and black hole solutions subsequently become more complicated. However, an interesting new feature arises called the attractor mechanism, which we should also try to understand.

We introduce the Schwarzschild black hole in section 3.2.1, which allows us to discuss some interesting concepts that are related to all black holes solutions, such as an event horizon, surface gravity, mass and thermodynamics. Here we also provide our working definition of a black hole solution. We then give the details of the Reissner–Nordström and Kerr–Newman solutions in section 3.2.2. We then take a first look at black hole solutions in supersymmetry and the attractor mechanism in section 3.2.3. This section is concluded with an example of a black hole solution to the *STU* model, which is a particular model of supergravity coupled to three vector multiplets.

3.2.1 Schwarzschild black hole and general properties

We begin our discussion of black hole physics by introducing the simplest black hole solution: the Schwarzschild black hole. Analysing the Schwarzschild metric leads naturally to a discussion of singularities and event horizons, which allows us to present our working definition of a black hole. We then discuss some more interesting properties of black holes in general, such as surface gravity, mass and thermodynamics. Since all the spherically symmetric black holes we will encounter in this thesis contain the Schwarzschild solution as a limit in which the charge and angular momentum vanish, it will also be of direct relevance later on in this thesis.

A minimal theory of pure gravity is described by the Einstein–Hilbert action³

$$S_{\text{EH}} = \frac{1}{16\pi G_{\text{N}}} \int d^4x \, \mathbf{e} \, R .$$

Gravity can be coupled minimally to some matter content S_{mat} using the action $S_{\text{EM}} + S_{\text{mat}}$ and replacing the partial derivatives appearing in S_{mat} with covariant derivatives. The energy-momentum tensor associated with the matter sector is given by

$$T_{\mu\nu} = -2\mathbf{e}^{-1} \frac{\delta S_{\text{mat}}}{\delta g^{\mu\nu}} .$$

The equations of motion corresponding to the variation of the whole action $S_{\text{EM}} + S_{\text{M}}$ with respect to the metric can then be written as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_{\text{N}}T_{\mu\nu} ,$$

which henceforth shall be called the Einstein equations. Vacuum solutions are by definition solutions to the Einstein equations with $T_{\mu\nu} = 0$. By taking the trace of the remaining Einstein equations we find that the Ricci scalar vanishes $R = 0$, and, hence, the Ricci tensor vanishes $R_{\mu\nu} = 0$.

The oldest solution of the vacuum Einstein equations is given by the Schwarzschild metric [90]

$$ds^2 = - \left(1 - \frac{2MG_{\text{N}}}{r} \right) dt^2 + \left(1 - \frac{2MG_{\text{N}}}{r} \right)^{-1} dr^2 + r^2 d\Omega_{(2)}^2 ,$$

³Recall that throughout this thesis we are using the notation $\mathbf{e} = \sqrt{-g}$.

where $d\Omega_{(2)}^2$ represents the line element on the unit two-sphere

$$d\Omega_{(2)}^2 = d\theta^2 + \sin^2 \theta d\phi^2 .$$

The most obvious features of the solution are that it is spherically symmetric and asymptotically Minkowski. In fact, by Birkhoff's theorem it is the unique spherically symmetric solution to the vacuum Einstein equations [91]. Another important feature of this solution is that it is stationary. This means that there exists a timelike Killing vector field⁴ ξ , i.e.

$$\mathcal{L}_\xi g = 0 .$$

Moreover, if we parametrise the flow by t and choose this to be our timelike coordinate then the metric is independent of t . The Schwarzschild metric actually satisfies the stronger condition of being static. This means the ξ defines an integrable distribution, i.e. it is normal to a set of hypersurfaces that foliate spacetime. In this case one can choose coordinates in which the mixed terms g_{ti} of the metric vanish.

The Schwarzschild solution appears to have a singularity in the metric at $r = 2MG_N$, but since this can be removed by a coordinate transformation it is not considered a true singularity of the underlying spacetime. On the other hand the point $r = 0$ is a true singularity in the sense that scalar curvature, which is independent of the choice of coordinates, is divergent. The choice of radial coordinate $r = 2MG_N$, which actually describes a sphere, is still physically interesting, however. It describes a surface called an event horizon, which can be understood as follows: consider an observer falling directly into a Schwarzschild black hole, i.e. their worldline is fixed in the θ, ϕ coordinates. The infinitesimal ratio between proper time τ and coordinate time t is given in terms of r by [92]

$$\frac{d\tau}{dt} = \left(1 - \frac{2MG_N}{r}\right)^{1/2} =: V(r) .$$

Now consider two static observers at positions $r_1 < r_2$ (with the same θ, ϕ coordinates), and let the observer at r_1 emit a light ray of frequency ω_1 outward towards the observer at position r_2 , who measures the frequency to be ω_2 . The two frequencies are related by

$$\frac{\omega_1}{\omega_2} = \frac{V(r_2)}{V(r_1)} < 1 .$$

The frequency of the light ray appears to the observer at r_2 to be shifted towards the red end of the spectrum, or 'red-shifted'. As the first observer moves closer toward the horizon the red-shift becomes infinite. On and beyond the horizon no light can escape to an outside observer. This motivates us to call such an object a black hole.

Definition 3.1 (black hole). A black hole metric is a solution of a classical theory containing gravity that satisfies the following properties:

- (i) It is asymptotically Minkowski.
- (ii) There exists an event horizon with finite area.

⁴In mathematical terminology a Killing vector field is simply a vector field that is an isometry of the metric.

In the literature one often comes across more refined definitions of a black hole, usually relating to the causal structure of spacetime. The definition presented above has the advantage of being simple and intuitive, and will be adequate for this thesis. If a metric admits a single event horizon then it describes a single black hole. We will later come across metrics which admit many separate horizons, and in which case describe multiple black holes. It is also worth mentioning that black hole metrics need not be stationary; however this is the only type of black hole solution we will encounter in this thesis.

Let us now continue with our analysis of the Schwarzschild black hole by calculating the ADM mass. Mass is a difficult property to define in general relativity, but in asymptotically flat spacetimes one can define a concrete notion of mass by integrating the gravitational field strength over a hypersurface shell of infinite radius. This is called the ADM mass, and is determined by the formula [93]

$$M_{\text{ADM}} = \frac{1}{16\pi G_{\text{N}}} \oint_{\Sigma_{\infty}} d^2\Sigma^i \left[\partial^j h_{ij} - \partial_i (\delta^{kl} h_{kl}) \right], \quad (3.11)$$

where h_{ij} is the purely spatial part of the metric, i.e. $ds^2 = g_{tt}dt^2 + 2g_{ti}dtdx^i + h_{ij}dx^i dx^j$. The submanifold Σ_{∞} is a hypersurface at spatial infinity, where $d\Sigma^i$ represents the line element. For example, the ADM mass of the Schwarzschild metric is given by

$$M_{\text{ADM}} = -\frac{1}{8\pi G_{\text{N}}} \lim_{r \rightarrow \infty} \oint_{\Sigma_r} d^2\Sigma^r \partial_r \left(1 - \frac{2MG_{\text{N}}}{r} \right)^{-1},$$

where the hypersurface $\Sigma_r = S_r$ is a two-sphere of radius r with line element

$$d^2\Sigma^r = \mathbf{e} \frac{1}{2} \epsilon_{rij} dx^i \wedge dx^j = r^2 \sin \theta d\theta \wedge d\varphi.$$

After performing the integration we find the expected result

$$M_{\text{ADM}} = -\frac{1}{2G_{\text{N}}} \lim_{r \rightarrow \infty} r^2 \partial_r \left(1 - \frac{2MG_{\text{N}}}{r} \right)^{-1} = M.$$

Another interesting property of a black hole is the surface gravity. Consider an observer at a constant radius $r > 2MG_{\text{N}}$. This observer will experience an acceleration due to gravity, which for the Schwarzschild black hole is given by

$$a^\mu = \nabla^\mu \log V(r) \quad \Rightarrow \quad |a| = \frac{\sqrt{\nabla_\mu V(r) \nabla^\mu V(r)}}{V(r)}.$$

At the horizon the denominator approaches zero, but the numerator $V(r)|a|$ remains finite. For the Schwarzschild black hole this takes the value

$$\kappa_{\text{S}} = \lim_{r \rightarrow 2MG_{\text{N}}} V(r)|a| = \frac{1}{4MG_{\text{N}}},$$

which we call the surface gravity of the black hole as it represents the strength of the gravitational field at the event horizon.

The macroscopic observables of a black hole system (mass M , charge Q and angular mom. J) satisfy the following laws of black hole mechanics [92]

0. The surface gravity κ_{S} is constant for a stationary black hole.

1. When comparing two infinitesimally close black holes the difference in mass is given by

$$\delta M = \frac{1}{8\pi} \kappa_S \delta A + \Omega \delta J + \mu \delta Q ,$$

where A is the area of the event horizon.

2. The area of the event horizon is strictly increasing $\delta A \geq 0$ for any process.
3. The surface gravity cannot vanish by any physical process.

These laws bear a striking resemblance to the laws of thermodynamics. The thermodynamical analogue of the zeroth law is the statement that the temperature is constant in a system in thermodynamic equilibrium. The first law of thermodynamics ensures the conservation of energy through the formula

$$\delta E = T \delta S + p \delta V + \mu \delta N ,$$

which is clearly similar to the first law of black hole mechanics. The second law of thermodynamics states that entropy can never decrease through any physical process for a closed system, as for the area of the event horizon. The analogue in thermodynamics of the surface gravity not vanishing is that the temperature never vanishes in a physical process, which completes the analogy for all four laws.

3.2.2 Reissner–Nordström and Kerr–Newman black holes

Let us now consider a theory of gravity coupled to the field strength of an abelian gauge vector, commonly called Einstein–Maxwell theory⁵

$$S_{\text{EM}} = \frac{1}{2} \int d^4x \, \mathbf{e} \left[R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] . \quad (3.12)$$

There exists a unique, static, spherically symmetric and purely electric solution to the field equations given by

$$ds^2 = - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right)^{-1} dr^2 + r^2 d\Omega_{(2)}^2 .$$

This is called the Reissner–Nordström solution, having been discovered independently by Reissner and Nordström [94, 95]. It is clear that in the limit $Q \rightarrow 0$ we obtain the Schwarzschild black hole. In this sense the Reissner–Nordström solution represents a controlled deviation away from the Schwarzschild black hole, where the deviation is measured by the electric charge Q . In order for an event horizon to exist the bound $M \geq |Q|$ must be respected. The particular solution that satisfies $M = |Q|$ is called an extremal Reissner–Nordström black hole, and exhibits many distinguished features, which we will come to shortly. The case where $M > Q$ is called a non-extremal black hole. In the case where $M < Q$ no horizon exists, and the solution exhibits a naked singularity.

It is useful to define $r_{\pm} = M \pm \sqrt{M^2 - Q^2}$ and $c = \frac{1}{2}(r_+ - r_-)$, in which case

$$\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) = \left(1 - \frac{r_+}{r} \right) \left(1 - \frac{r_-}{r} \right) .$$

⁵From now on we set $8\pi G_N = 1$.

The extremal limit is given by $c \rightarrow 0$, and for this reason c is called the non-extremality parameter. In the extremal limit $r_+ \rightarrow r_-$ and both horizons degenerate into just one horizon. A particularly useful parametrisation of the Reissner–Nordström line element is obtained by introducing the shifted radial coordinate $\rho = r - r_-$, in which case

$$\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) = \frac{W}{\mathcal{H}^2},$$

where

$$W = 1 - \frac{2c}{\rho}, \quad \mathcal{H} = 1 + \frac{r_-}{\rho}.$$

The full Reissner–Nordström line element can then be written as

$$ds^2 = -\frac{W}{\mathcal{H}^2} dt^2 + \mathcal{H}^2 \left(\frac{d\rho^2}{W} + \rho^2 d\Omega_{(2)}^2 \right). \quad (3.13)$$

In the extremal limit $W \rightarrow 1$, which shows that a non-extremal black hole can be obtained from an extremal black hole by ‘dressing’ the line element with an additional harmonic function W . The extremal Reissner–Nordström metric is written as

$$ds^2 = -\mathcal{H}^{-2} dt^2 + \mathcal{H}^2 (\delta_{ij} dx^i dx^j), \quad (3.14)$$

where $\mathcal{H} = 1 - \frac{M}{\rho}$ with M the mass of the black hole.

Non-extremal Reissner–Nordström black holes have an outer event horizon located at $r = r_+$, which, as we have previously discussed, is a hypersurface of infinite red shift. They also exhibit a second type of horizon, known as a Cauchy horizon, located at $r = r_- < r_+$. This inner horizon is a hypersurface of infinite blue shift, and has a repulsive surface gravity, equal in magnitude to the surface gravity at the outer horizon. Again, there is a singularity located at $r = 0$. In the extremal limit both horizons coincide, and the surface gravity vanishes. For such black holes the attractive force of gravity exactly balances with the repulsive force of electromagnetism. Therefore two extremal black holes will neither attract nor repel one another, which is known as the ‘no-force’ property.

In order to solve the static field equations it is not actually necessary that the metric is spherically symmetric. A class of non-spherically symmetric solutions is given by taking the line element (3.14) but requiring only that \mathcal{H} is a harmonic function with respect to the flat three-dimensional metric

$$\Delta \mathcal{H} = 0.$$

These are known as the Majumdar–Papapetrou class of solutions. However, the only type of solutions that do not exhibit naked singularities are the multi-centred generalisations of the extremal Reissner–Nordström solution [96]

$$\mathcal{H} = 1 + \sum_{\alpha} \frac{M_{\alpha}}{|\vec{\rho} - \vec{\rho}_{\alpha}|},$$

which describes multiple black holes located at $\vec{\rho} = \vec{\rho}_{\alpha}$. This solution describes a configuration of multiple extremal black holes with individual masses M_{α} , which remain in static equilibrium due to the no-force property of the extremal black holes.

Let us consider again solutions to the Einstein–Maxwell theory. We will relax the condition that solutions are static and spherically symmetric, and demand only that

solutions are stationary and axially symmetric. In this case there exists a unique black hole solution that is characterised by mass M , charge Q and angular momentum J . This is given by the Kerr–Newman metric [97]

$$ds^2 = -\frac{\Delta}{R}(dt - \alpha \sin^2 \theta d\varphi)^2 + R \left(\frac{dr^2}{\Delta} + d\theta^2 \right) + \frac{\sin^2 \theta}{R} ((r^2 + \alpha^2)d\varphi - \alpha dt)^2 ,$$

where $\alpha = \frac{J}{M}$ is a constant parameter and

$$\begin{aligned} R &= r^2 + \alpha^2 \cos^2 \theta , \\ \Delta &= r^2 - 2Mr + \alpha^2 + Q^2 . \end{aligned}$$

Clearly in the static limit $\alpha \rightarrow 0$ we recover the Reissner–Nordström solution. One can also consider the limit $Q \rightarrow 0$, in which case we have a rotating uncharged black hole, called simply a Kerr black hole. We are using a set of oblate spheroidal coordinates (a.k.a. Boyer–Lindquist coordinates), which are related to cartesian coordinates through

$$\begin{aligned} x &= \sqrt{r^2 + \alpha^2} \sin \theta \cos \varphi , \\ y &= \sqrt{r^2 + \alpha^2} \sin \theta \sin \varphi , \\ z &= r \cos \theta . \end{aligned}$$

In the static limit these reduce to the standard spherical polar coordinates.

For a Kerr–Newman black hole an event horizon only exists if the bound $M^2 \geq Q^2 + J^2$ is satisfied. The event horizon will not be a sphere, but a so-called oblate spheroid. The singularity will no longer be a point but a ring, located at $r = \alpha \sin \theta$. For non-zero J the particular solution for which $M = |Q|$ will have a naked singularity and therefore cannot be called a black hole. However, for this over-rotating class of solutions the three-dimensional part of the metric is flat. This will play an important role later in the thesis. For later reference, the Kerr–Newman metric can be written in a manner adapted to dimensional reduction as

$$ds^2 = -e^\phi(dt^2 + V_\varphi d\varphi)^2 + e^{-\phi} ds_3^2 , \quad (3.15)$$

where

$$\begin{aligned} e^\phi &= \frac{\Delta - \alpha^2 \sin^2 \theta}{\Sigma} , \\ V_\varphi &= \frac{\alpha \sin^2 \theta (r^2 + \alpha^2 - \Delta)}{\Delta - \alpha^2 \sin^2 \theta} , \\ ds_3^2 &= \frac{\Delta - \alpha^2 \sin^2 \theta}{\Delta} dr^2 + (\Delta - \alpha^2 \sin^2 \theta) d\theta^2 + \Delta \sin^2 \theta d\varphi^2 . \end{aligned}$$

We end our discussion of black hole solutions of the Einstein–Maxwell theory by remarking that there exist higher-dimensional analogues of such solutions [98]. In higher dimensions there also exist black objects with more exotic topologies, but none of which will be directly relevant to this thesis. What will be relevant is the five-dimensional analogue of the Reissner–Nordström solution. This simply corresponds to adjusting certain numerical factors in the metric, and setting the poles of the harmonic functions to scale as $1/\rho^2$. We refer the reader to [60] for a more detailed overview.

3.2.3 Black holes in supergravity

The bosonic part of the action of pure $4d$, $\mathcal{N} = 2$ supergravity is precisely the same as the Einstein–Maxwell Lagrangian (3.12). Therefore, we can embed the Reissner–Nordström solution into a theory of pure supergravity by adding two gravitini ψ_μ^A to the action. The Reissner–Nordström solution is then interpreted as a bosonic background solution, i.e. $\psi_\mu^A = 0$.

For the extremal Reissner–Nordström solution the embedding into supergravity allows us to understand its peculiar features. This is because there exist certain supersymmetry transformations under which the solutions is invariant [99]

$$\delta_{\epsilon(x)}(g_{\mu\nu}, \psi_\mu^A, A_\mu) \Big|_{\text{extRN}} = 0 .$$

The fermionic parameters ϵ are analogous to Killing vectors, and are therefore referred to as Killing spinors and the above equations as ‘Killing spinor equations’. The extremal Reissner–Nordström black hole admits precisely four Killing spinors, and is consequently invariant under half the supersymmetry transformations. This suggests that we should interpret the extremal Reissner–Nordström solution as a BPS solution of pure supergravity. This identification is given more support by the fact that the Reissner–Nordström solution satisfies the mass bound $M = |Q|$, which is analogous to the mass bound for BPS states $M = |Z|$. There are many other features that the Reissner–Nordström exhibits that are analogous to BPS states, and so the identification is understood to hold concretely [36]. More generally one can identify the Majumdar–Papapetrou solutions as BPS states [37]. The central charge generates a local $U(1)$ phase symmetry, with the graviphoton as the gauge vector, and can be written as [100]

$$Z = \frac{1}{2\pi} \int F^- = P - iQ ,$$

where F^- is the anti-self-dual part of the graviphoton field strength. It is clear that for electric solutions we have $|Z| = |Q|$.

The Kerr–Newman solution may also be embedded into a theory of pure supergravity. However in this case the BPS bound (invariant under half supersymmetry transformations) and the extremality bound (horizon disappears) are not the same. In fact the BPS bound is stronger than the extremality bound, so BPS black holes are said to be ‘over-rotating’ and have a naked singularity [101]. They necessarily have a flat three-dimensional metric according to the decomposition given by (3.15).

Coupling vector multiplets to pure supergravity introduces a number of new scalar and vector fields into the action according to (3.5), as previously discussed. One can find many extremal black hole solutions analogous to the extremal Reissner–Nordström black hole that can also be interpreted as BPS states [7]. However, these solutions will generically not depend just on the electric and magnetic charges of the various gauge fields, but will also contain contributions from the scalar fields. In this case one can define a useful object, also called the central charge, by

$$Z = P^I F_I - Q_I X^I .$$

This is a spacetime dependent function that is equal to the central charge of the supersymmetry algebra when evaluated at spatial infinity, and gives the entropy of a BPS black hole when evaluated on the horizon

$$S = \frac{1}{4} A = \pi |Z|_{\text{hor}}^2 . \quad (3.16)$$

The values of the scalar fields at the horizon are completely determined by the charges through the expressions

$$\left[\bar{Z} \begin{pmatrix} X^I \\ F_I \end{pmatrix} - Z \begin{pmatrix} \bar{X}^I \\ \bar{F}_I \end{pmatrix} \right]_{\text{hor}} = i \begin{pmatrix} P^I \\ Q_I \end{pmatrix}. \quad (3.17)$$

This is known as the attractor mechanism [7, 39, 46], since as a dynamical system the flow of the scalar fields exhibits fixed point behaviour which is attractive. The equations themselves are called attractor equations or stabilisation equations, and follow directly from the Killing spinor equations [39, 40]. Since the scalar fields are fixed in terms of the charges on the horizon, the entropy of the black hole (3.16) is completely determined by the charges.

As an example let us consider the STU model, which can be found in e.g. [102]. We will come back to this model at several occasions later in this thesis. The STU model consists of three vector multiplets coupled to supergravity, and is characterised by the prepotential

$$F = -\frac{X^1 X^2 X^3}{X^0}.$$

The physical scalar fields are given by $z^A = X^A/X^0$ and are often denoted by $S = z^1$, $T = z^2$, $U = z^3$ (which is where the model gets its name). We will consider solutions in which X^A are purely imaginary and X^0 is purely real. In this case the static BPS solution for the metric and scalar fields is given by

$$ds^2 = -\frac{1}{\sqrt{-4\mathcal{H}_0\mathcal{H}^1\mathcal{H}^2\mathcal{H}^3}} dt^2 + \sqrt{-4\mathcal{H}_0\mathcal{H}^1\mathcal{H}^2\mathcal{H}^3} (\delta_{ij} dx^i dx^j), \quad (3.18)$$

$$X^0 = (-4\mathcal{H}_0\mathcal{H}^1\mathcal{H}^2\mathcal{H}^3)^{\frac{1}{4}}, \quad X^A = i\mathcal{H}^A\mathcal{H}_0 (-4\mathcal{H}_0\mathcal{H}^1\mathcal{H}^2\mathcal{H}^3)^{-\frac{1}{4}},$$

where $\mathcal{H}_0, \mathcal{H}^A$ are harmonic functions

$$\mathcal{H}_0 = h_0 + \frac{Q_0}{\rho}, \quad \mathcal{H}^A = h^A + \frac{P^A}{\rho}.$$

Taking the near horizon limit $\rho \rightarrow 0$ the scalar fields clearly depend only on the charges, and we observe attractor behaviour

$$z^A = \frac{i\mathcal{H}^A\mathcal{H}_0}{\sqrt{-4\mathcal{H}_0\mathcal{H}^1\mathcal{H}^2\mathcal{H}^3}} \longrightarrow \frac{iP^AQ_0}{\sqrt{-4Q_0P^1P^2P^3}}.$$

The entropy of the black hole depends only on the charges and not on the value of the moduli fields at infinity

$$S = \frac{1}{4}A = \pi\sqrt{-4Q_0P^1P^2P^3} = |Z|_{\text{hor}}^2.$$

In the limit where the four harmonic functions are proportional to one another the solution reduces to the standard extremal Reissner–Nordström black hole, which can be seen by directly comparing the line element (3.14) with (3.18). The ADM mass is given by

$$M = \frac{Q_0}{4h_0} - 3h_0(p^1h^2h^3 + p^2h^1h^3 + p^3h^1h^2) = |X^IQ_I - F_IP^I|_{\infty} = |Z|_{\infty},$$

and so the BPS bound is satisfied. One may also check that the solution satisfies the attractor equations (3.17).

3.3 Dimensional Reduction

Classically dimensional reduction amounts to requiring that all fields are invariant under the flow generated by some vector field $\frac{\partial}{\partial x^0}$, which has orbits given by S^1 . Dimensional reduction by integrating out the x^0 dimension simply gives the space of orbits. In a quantum theory a massless field may be Fourier expanded in a compact dimension into a tower of massive fields, where the masses depend inversely on the size of the compact dimension. In this case one may instead require that the compact dimension is small enough such that any massive modes can be ignored in an effective theory. In either case, the dimensional reduction procedure is the same, which we will now present in detail. These calculations are based largely on a set of unpublished notes by Ulrich Theis.

We will dimensionally reduce a theory of gravity coupled to the field strength of a p -form gauge field over a single compact timelike ($\epsilon = +1$) or spacelike ($\epsilon = -1$) dimension of radius λ . The main results of this section are as follows:

1. The Einstein–Hilbert action

$$\hat{S}_{\text{EH}} = \int d^{(n+1)}x \, \hat{\mathbf{e}} \left[\frac{1}{2} \hat{R} \right],$$

reduces to

$$S_{\text{EH}} = 2\pi\lambda \int d^n x \, \mathbf{e} \left[\frac{1}{2} R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \epsilon e^{\left(\frac{2n-2}{n-2}\right)^{1/2} \phi} V_{\mu\nu} V^{\mu\nu} \right], \quad (3.19)$$

where we have made the metric decomposition

$$\hat{g} = -\epsilon e^{\left(\frac{2(n-2)}{n-1}\right)^{\frac{1}{2}} \phi} (dx^0 + V_\mu dx^\mu)^2 + e^{-\left(\frac{2}{(n-2)(n-1)}\right)^{\frac{1}{2}} \phi} g_{\mu\nu} dx^\mu dx^\nu. \quad (3.20)$$

2. The action of the field strength of an abelian gauge vector coupled to gravity

$$\hat{S}_{\text{gauge}} = \int d^{(n+1)}x \, \hat{\mathbf{e}} \left[-\frac{1}{4} \hat{F}_{\hat{\mu}\hat{\nu}} \hat{F}^{\hat{\mu}\hat{\nu}} \right],$$

reduces to

$$S_{\text{gauge}} = 2\pi\lambda \int d^n x \, \mathbf{e} \left[\epsilon e^{-\left(\frac{2(n-2)}{n-1}\right)^{\frac{1}{2}} \phi} \frac{1}{2} \partial_\mu b \partial^\mu b - \frac{1}{4} e^{\left(\frac{2}{(n-2)(n-1)}\right)^{\frac{1}{2}} \phi} (F_{\mu\nu} - 2V_{[\mu} \partial_{\nu]} b) (F^{\mu\nu} - 2V^{[\mu} \partial^{\nu]} b) \right]. \quad (3.21)$$

where we have made the decomposition

$$\hat{A} = b dx^0 + A_\mu dx^\mu. \quad (3.22)$$

3. In four dimensions we may also include the topological term in the gauge sector

$$\hat{S}_{\text{top}} = \int d^4 x \left[\frac{1}{8} \hat{F}_{\hat{\mu}\hat{\nu}} \hat{F}_{\hat{\rho}\hat{\sigma}} \hat{\epsilon}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \right],$$

which, under the decomposition given above, reduces to

$$S_{\text{top}} = 2\pi\lambda \int d^3 x \left[\frac{1}{2} \epsilon F_{\mu\nu} \partial_\rho b \epsilon^{\mu\nu\rho} \right]. \quad (3.23)$$

3.3.1 The action and differential forms

Consider a Riemannian manifold (M, g) of dimension m and metric signature (f, g) . We will construct an action functional $S = S_{\text{EH}} + S_{\text{gauge}}$, which consists of an Einstein Hilbert term coupled to the field strength of a p -form gauge field, in terms of differential forms. Our aim is to recover the standard form of the action when written in components

$$S_{\text{EH}} = \int d^m x \, \mathbf{e} \left[\frac{1}{2} R \right] ,$$

$$S_{\text{gauge}} = \int d^m x \, \mathbf{e} \left[\frac{-1}{2(p+1)!} (F_{p+1})_{\mu_1 \dots \mu_{p+1}} (F_{p+1})^{\mu_1 \dots \mu_{p+1}} \right] .$$

Note that we are using a non-standard normalisation of the Einstein–Hilbert term (as is common in the supergravity literature) in order to avoid factors of $\sqrt{2}$ at a later stage.

Let us first construct the Einstein–Hilbert term. Since the Ricci scalar R is simply a function, the Hodge star is given by

$$\star R = \frac{\mathbf{e}}{m!} R \epsilon_{\mu_1 \dots \mu_m} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_m} = (-)^f \mathbf{e} R dx^1 \wedge \dots \wedge dx^m .$$

We can therefore write

$$S_{\text{EH}} = (-)^f \int \frac{1}{2} \star R .$$

It will be useful later to note that

$$\begin{aligned} \epsilon_{a_1 \dots a_m} R^{a_1 a_2} \wedge e^{a_3} \wedge \dots \wedge e^{a_m} &= \epsilon_{a_1 \dots a_m} \left(\frac{1}{2} R^{a_1 a_2}{}_{bc} e^b \wedge e^c \right) \wedge e^{a_3} \wedge \dots \wedge e^{a_m} \\ &= (-)^f (m-2)! R e^1 \wedge \dots \wedge e^m , \end{aligned}$$

and therefore

$$\star R = \frac{1}{(m-2)!} \epsilon_{a_1 \dots a_m} R^{a_1 a_2} \wedge e^{a_3} \wedge \dots \wedge e^{a_m} . \quad (3.24)$$

Next we will focus on the field strength term. The field strength of a p -form A_p is obtained by taking the exterior derivative

$$F_{p+1} = dA_p = \frac{1}{(p+1)!} (F_{p+1})_{\mu_1 \dots \mu_{p+1}} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{p+1}} .$$

The Hodge dual of F_{p+1} is an $(m-p-1)$ -form

$$\star F_{p+1} = \frac{\mathbf{e}}{(p+1)!(m-p-1)!} (F_{p+1})_{\nu_1 \dots \nu_{p+1}} \epsilon^{\nu_1 \dots \nu_{p+1}}{}_{\rho_{p+2} \dots \rho_m} dx^{\rho_{p+2}} \wedge \dots \wedge dx^{\rho_m} .$$

Taking the wedge product of F_{p+1} and $\star F_{p+1}$ we find

$$\begin{aligned} F_{p+1} \wedge \star F_{p+1} &= \frac{\mathbf{e}}{(p+1)!^2 (m-p-1)!} (F_{p+1})_{\mu_1 \dots \mu_{p+1}} (F_{p+1})_{\nu_1 \dots \nu_{p+1}} \\ &\quad \times \epsilon^{\nu_1 \dots \nu_{p+1}}{}_{\rho_{p+2} \dots \rho_m} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{p+1}} \wedge dx^{\rho_{p+2}} \wedge \dots \wedge dx^{\rho_m} \\ &= (-)^f \frac{\mathbf{e}}{(p+1)!} (F_{p+1})_{\mu_1 \dots \mu_{p+1}} (F_{p+1})^{\mu_1 \dots \mu_{p+1}} dx^1 \wedge \dots \wedge dx^m . \end{aligned}$$

We can now construct the S_{gauge} term in our action

$$S_{\text{gauge}} = (-)^f \int -\frac{1}{2} F_{p+1} \wedge \star F_{p+1} .$$

3.3.2 Decomposing the metric

We will now proceed to reduce an action of gravity coupled to a p -form from $(n+1)$ dimensions to n dimensions. The compact dimension is taken to be a circle S^1 , and can be a spacelike or timelike dimension which we keep track of by setting

$$\epsilon = \begin{cases} +1 & \text{timelike ,} \\ -1 & \text{spacelike .} \end{cases}$$

Objects that live in $(n+1)$ dimensions will be distinguished by a hat, where hatted indices run from $0, \dots, n$ and unhatted indices from $1, \dots, n$, e.g.

$$\begin{aligned} (n+1)\text{-dimensional vielbeins } \hat{e}^{\hat{a}}, \quad \hat{a} \in \{0, \dots, n\}, \\ n\text{-dimensional vielbeins } e^a, \quad a \in \{1, \dots, n\}. \end{aligned}$$

The signature of the $(n+1)$ -dimensional metric in the transverse dimensions may be arbitrary, but for concreteness let us say there are f timelike and g spacelike directions, so we can write the signature of the $(n+1)$ -dimensional metric as

$$(-\underbrace{\epsilon \dots \epsilon}_{f\text{-times}} \underbrace{+ \dots +}_{g\text{-times}}).$$

We take the compact dimension to be x^0 .

Let us begin by decomposing the metric into various n -dimensional components

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \left(\begin{array}{c|c} -\epsilon e^{2\beta\phi} & -\epsilon e^{2\beta\phi} V_\mu \\ \hline -\epsilon e^{2\beta\phi} V_\nu & e^{-2\alpha\phi} g_{\mu\nu} - \epsilon e^{2\beta\phi} V_\mu V_\nu \end{array} \right), \quad (3.25)$$

where α and β are, for now, arbitrary numbers with $\beta \neq 0$. It will be useful to translate this decomposition into a vielbein basis, which is defined by the transformation matrices $\hat{e}_{\hat{\mu}}^{\hat{a}}$ satisfying

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \hat{e}_{\hat{\mu}}^{\hat{a}} \hat{e}_{\hat{\nu}}^{\hat{b}} \hat{\eta}_{\hat{a}\hat{b}}.$$

First, by considering the indices $\hat{\mu} = \mu$, $\hat{\nu} = \nu$ we obtain the expressions

$$e^{-2\alpha\phi} g_{\mu\nu} - \epsilon e^{2\beta\phi} V_\mu V_\nu = -\epsilon \hat{e}_\mu^0 \hat{e}_\nu^0 + \hat{e}_\mu^a \hat{e}_\nu^b \eta_{ab},$$

and we have the freedom to choose the particular vielbein basis given by $\hat{e}_\mu^a = e^{-\alpha\phi} e_\mu^a$ and $\hat{e}_\mu^0 = e^{\beta\phi} V_\mu$. Next, by considering the indices $\hat{\mu} = \mu$, $\hat{\nu} = 0$ we obtain the expression

$$-\epsilon e^{2\beta\phi} V_\mu = e^{-\alpha\phi} e_\mu^a \hat{e}_0^b \eta_{ab} - \epsilon e^{\beta\phi} V_\mu \hat{e}_0^0,$$

and we can choose $\hat{e}_0^b = 0$ and $\hat{e}_0^0 = e^{\beta\phi}$. We have now completely determined the transformation matrices

$$\hat{e}_{\hat{\mu}}^{\hat{a}} = \left(\begin{array}{c|c} e^{\beta\phi} & 0 \\ \hline e^{\beta\phi} V_\mu & e^{-\alpha\phi} e_\mu^a \end{array} \right).$$

From this expression one can immediately obtain formulae that relate the n - and $(n+1)$ -dimensional vielbeins

$$\hat{e}^a = e^{-\alpha\phi} e^a, \quad \hat{e}^0 = e^{\beta\phi} (V + dx^0).$$

Moreover, one also obtains the formula relating the determinant of the n - and $(n+1)$ -dimensional metric

$$\det \hat{g} = -\epsilon e^{2(\beta-\alpha)\phi} \det g.$$

Accordingly, we define the epsilon symbol in n -dimensions to be

$$\epsilon_{12\dots n} = -\epsilon \hat{\epsilon}_{012\dots n}.$$

The inverse transformation matrices $\hat{e}_a^{\hat{\nu}}$ are uniquely determined through the expression $\hat{e}_{\hat{\mu}}^{\hat{a}} \hat{e}_a^{\hat{\nu}} = \delta_{\hat{\mu}}^{\hat{\nu}}$. For our particular choice of transformation matrices we find

$$\hat{e}_a^{\hat{\mu}} = \left(\begin{array}{c|c} e^{-\beta\phi} & 0 \\ \hline -e^{\alpha\phi} V_a & e^{\alpha\phi} e_a^{\mu} \end{array} \right).$$

The vector fields dual to the vielbein one-forms are then calculated to be

$$\hat{e}_a = e^{\alpha\phi} e_a - e^{\alpha\phi} V_a \frac{\partial}{\partial x^0}, \quad \hat{e}_0 = e^{-\beta\phi} \frac{\partial}{\partial x^0}.$$

3.3.3 Reduction of Einstein–Hilbert term

The dimensional reduction of the Einstein–Hilbert term is greatly simplified if we make the specific choices $\alpha = 0$ and $\beta = 1$, so that

$$\begin{aligned} \hat{e}^0 &= e^{\phi} (V + dx^0), & \hat{e}^a &= e^a, \\ \hat{e}_a &= e_a - V_a \frac{\partial}{\partial x^0}, & \hat{e}_0 &= e^{-\phi} \frac{\partial}{\partial x^0}. \end{aligned}$$

After dimensional reduction the metric will not be in the so-called ‘Einstein frame’, which means we will have a non-constant factor in front of the Einstein–Hilbert term in the action. However, one can always make a conformal transformation to bring the metric to the Einstein frame. We will then be able to read off the values of α and β that we could have chosen in order to reduce directly into this frame.

Recall from section 2.1.1 that the curvature two-form is given by

$$\hat{R}_{\hat{a}\hat{b}} = d\hat{\omega}_{\hat{a}\hat{b}} + \hat{\omega}_{\hat{a}}^{\hat{c}} \wedge \hat{\omega}_{\hat{c}\hat{b}}.$$

We can decompose this into n -dimensional objects using the following expression for the Levi-Civita one-form:

$$\hat{\omega}_{\hat{a}\hat{b}} = \frac{1}{2} \left(\hat{\eta}_{\hat{b}\hat{c}} \iota_{\hat{e}_{\hat{a}}} - \hat{\eta}_{\hat{a}\hat{c}} \iota_{\hat{e}_{\hat{b}}} - \hat{\eta}_{\hat{c}\hat{d}} \hat{e}^{\hat{d}} \iota_{\hat{e}_{\hat{a}}} \iota_{\hat{e}_{\hat{b}}} \right) d\hat{e}^{\hat{c}}.$$

Considering the indices $\hat{a} = a$, $\hat{b} = b$ in the above expression we find

$$\hat{\omega}_{ab} = \omega_{ab} - \frac{1}{2} \epsilon e^{2\phi} (V + dx^0) V_{ab},$$

and by considering the indices $\hat{a} = a$, $\hat{b} = 0$ we also have

$$\hat{\omega}_{a0} = -\frac{1}{2}\epsilon e^\phi(\iota_{e_a}dV) + \epsilon e^\phi(V + dx^0)(\iota_{e_a}d\phi) .$$

Taking the exterior derivative of $\hat{\omega}_{ab}$ we find

$$d\hat{\omega}_{ab} = d\omega_{ab} - \frac{1}{2}\epsilon e^{2\phi}V_{ab}dV + \mathcal{O}(V + dx^0) ,$$

where we have ignored terms containing the combination $(V + dx^0)$ for reasons that will become clear shortly. Next, we must decompose the term

$$\hat{\omega}_a^{\hat{c}} \wedge \hat{\omega}_{\hat{c}b} = \omega_a^c \wedge \omega_{cb} + \frac{1}{4}\epsilon e^{2\phi}(\iota_{e_a}dV) \wedge (\iota_{e_b}dV) + \mathcal{O}(V + dx^0) .$$

Putting these together we obtain the following decomposition of the Riemann curvature tensor

$$\hat{R}_{ab} = R_{ab} - \frac{1}{2}\epsilon e^{2\phi}(V_{ab}dV - \frac{1}{2}(\iota_{e_a}dV) \wedge (\iota_{e_b}dV)) + \mathcal{O}(V + dx^0) .$$

We now repeat the process considering the indices $\hat{a} = a$, $\hat{b} = 0$ in the expression for the Riemann curvature tensor. The exterior derivative of the $\hat{\omega}_{a0}$ components of the Levi-Civita one-form are given by

$$d\hat{\omega}_{a0} = \epsilon d\phi \wedge e^\phi(V + dx^0)(\iota_{e_a}d\phi) - \epsilon e^\phi(V + dx^0) \wedge d((\iota_{e_a}d\phi)) + \dots ,$$

where we have ignored terms which do not contain the combination $(V + dx^0)$. Again, we must decompose the term

$$\hat{\omega}_a^{\hat{c}} \wedge \hat{\omega}_{\hat{c}0} = \frac{1}{4}e^{3\phi}(V + dx^0) \wedge V_a^c(\iota_{e_c}dV) + \epsilon e^\phi\omega_a^c \wedge (V + dx^0)(\iota_{e_c}d\phi) + \dots ,$$

where in the first line we have used the fact that $\hat{\omega}_{00} = 0$. Putting these together we find an expression for the 0th row (or column) of the Riemann curvature

$$\begin{aligned} \hat{R}_{a0} = \epsilon e^\phi \left[d\phi(\iota_{e_a}d\phi) + d((\iota_{e_a}d\phi)) - \omega_a^c(\iota_{e_c}d\phi) - \frac{1}{4}\epsilon e^{2\phi}V_a^c(\iota_{e_c}dV) \right] \\ \wedge (V + dx^0) + \dots . \end{aligned}$$

Recall that we can use (3.24) to write the Einstein–Hilbert action in terms of the curvature two-form. This expression can be decomposed as follows:

$$\begin{aligned} \hat{S}_{\text{EH}} = \int \frac{-1}{2(n-1)!} \hat{\epsilon}_{0a_1\dots a_n} \left[2\hat{R}^{0a_1} \wedge e^{a_2} + (n-1)e^\phi \hat{R}^{a_1a_2}(V + dx^0) \right] \\ \times e^{a_3} \wedge \dots \wedge e^{a_n} , \end{aligned}$$

and substituting in the expression for $\hat{R}^{a_1a_2}$ and \hat{R}^{a_10} we have

$$S_{\text{EH}} = \int dx^0 \int -\epsilon(-)^f \left[e^\phi \frac{1}{2} {}^\star R - \frac{1}{2}\epsilon e^{3\phi} dV \wedge {}^\star dV \right] - 2\epsilon(-)^f \int d {}^\star de^\phi .$$

Observe that the $d\phi$ term forms a total derivative which we can drop. We now integrate out the compact dimension by performing the finite integral

$$\int dx^0 = \frac{8}{16} \frac{G_N^{(n+1)}}{G_N^n} .$$

We can make a conformal rescaling in order to write the action in the so-called ‘Einstein frame’, where the Einstein–Hilbert action takes the standard form. The conformal transformation we need is

$$g_{\mu\nu} \longrightarrow e^{\left(\frac{-2}{n-2}\right)\phi} g_{\mu\nu} ,$$

so that the action becomes

$$S_{\text{EH}} = \int -\epsilon (-)^f \left[\frac{1}{2} {}^\star R - \left(\frac{n-1}{n-2} \right) d\phi \wedge {}^\star d\phi + \frac{1}{2} \epsilon e^{\left(\frac{2n-2}{n-2}\right)\phi} dV \wedge {}^\star dV \right] .$$

Finally, we can properly normalise the kinetic term of the scalar field ϕ by making the rescaling

$$\phi \longrightarrow \sqrt{\frac{n-2}{2n-2}} \phi ,$$

and we are left with the final form of the dimensionally reduced Einstein–Hilbert term

$$S_{\text{EH}} = \int -\epsilon (-)^f \left[\frac{1}{2} {}^\star R - \frac{1}{2} d\phi \wedge {}^\star d\phi + \frac{1}{2} \epsilon e^{\left(\frac{2n-2}{n-2}\right)^{1/2} \phi} dV \wedge {}^\star dV \right] .$$

We can now read off the values for α and β that we could have chosen in (3.25) in order to reduce directly into the Einstein frame

$$\alpha = \frac{1}{\sqrt{2(n-1)(n-2)}} , \quad \beta = (n-2)\alpha .$$

3.3.4 Reduction of field strength term

An arbitrary p -form in $(n+1)$ dimensions can be written as

$$\hat{u}_p = \frac{1}{p!} (\hat{u}_p)_{\hat{\mu}_1 \dots \hat{\mu}_p} dx^{\hat{\mu}_1} \wedge \dots \wedge dx^{\hat{\mu}_p} .$$

The one-form dx^0 may appear at most once in each term. Splitting the expression into the sum of terms with dx^0 and those without, and using the symmetry to move dx^0 to the left, we can write

$$\hat{u}_p = \frac{1}{p!} (\hat{u}_p)_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} + \frac{p}{p!} (\hat{u}_p)_{0\nu_1 \dots \nu_{p-1}} dx^0 \wedge dx^{\nu_1} \wedge \dots \wedge dx^{\nu_{p-1}} ,$$

and defining

$$(u_p)_{\mu_1 \dots \mu_p} := (\hat{u}_p)_{\mu_1 \dots \mu_p} , \quad (u_{p-1})_{\mu_1 \dots \mu_{p-1}} := (\hat{u}_p)_{0\mu_1 \dots \mu_{p-1}} ,$$

we have the following decomposition of a p -form

$$\hat{u}_p = u_p + dx^0 \wedge u_{p-1} . \tag{3.26}$$

We will now consider the Hodge star of \hat{u}_p . This time it is easier to use a vielbein basis, in which we can write the Hodge star as

$$\hat{\star} \hat{u}_p = \frac{1}{p!(n+1-p)!} (\hat{u}_p)_{\hat{a}_1 \dots \hat{a}_p} \hat{\epsilon}^{\hat{a}_1 \dots \hat{a}_p}_{\hat{b}_{p+1} \dots \hat{b}_{n+1}} \hat{e}^{\hat{b}_{p+1}} \wedge \dots \wedge \hat{e}^{\hat{b}_{n+1}} .$$

Precisely one of the indices of $\hat{\epsilon}^{\hat{a}_1 \dots \hat{a}_p}_{\hat{b}_{p+1} \dots \hat{b}_{n+1}}$ must be 0, and so we can write this as two distinct sums

$$\begin{aligned} \star \hat{u}_p &= \frac{(n+1-p)}{p!(n+1-p)!} (\hat{u}_p)_{a_1 \dots a_p} \hat{\epsilon}^{a_1 \dots a_p}_{0b_{p+1} \dots b_n} \hat{e}^0 \wedge \hat{e}^{b_{p+1}} \wedge \dots \wedge \hat{e}^{b_n} \\ &+ \frac{p}{p!(n+1-p)!} (\hat{u}_p)_{0a_1 \dots a_{p-1}} \hat{\epsilon}^{0a_1 \dots a_{p-1}}_{b_p \dots b_n} \hat{e}^{b_p} \wedge \dots \wedge \hat{e}^{b_n} . \end{aligned} \quad (3.27)$$

Lets now consider the top and bottom lines separately.

First of all we consider the top line. Note that currently the components $(\hat{u}_p)_{\hat{a}_0 \dots \hat{a}_{p-1}}$ are given with respect to the $(n+1)$ -dimensional vielbein basis \hat{e}^a . We want to convert these to components of the n -dimensional vielbein basis e^a . We compute the components in the n -dimensional vielbein basis as follows

$$\begin{aligned} (\hat{u}_p)_{a_1 \dots a_p} A^{a_1 \dots a_p} &= \left[\hat{e}_{a_1}^{\hat{\mu}_1} \dots \hat{e}_{a_p}^{\hat{\mu}_p} (\hat{u}_p)_{\hat{\mu}_1 \dots \hat{\mu}_p} \right] A^{a_1 \dots a_p} \\ &= \left[-p e^{p\alpha\phi} V_{a_1} (u_{p-1})_{a_2 \dots a_p} + e^{p\alpha\phi} (u_p)_{a_1 \dots a_p} \right] A^{a_1 \dots a_p} , \end{aligned}$$

where $A^{a_1 \dots a_p}$ is some anti-symmetric tensor. Here we have used the fact that all the indices appear contracted with an antisymmetric tensor $A^{a_1 \dots a_p}$ so we can safely move the 0-index to the left. We then need the fact that

$$\begin{aligned} \hat{\epsilon}^{a_1 \dots a_p}_{0b_{p+1} \dots b_n} &= (-)^p \hat{\epsilon}_0^{a_1 \dots a_p}_{b_{p+1} \dots b_n} \\ &= -\epsilon (-)^p \epsilon^{a_1 \dots a_p}_{b_{p+1} \dots b_n} . \end{aligned}$$

The wedge product of the vielbeins can be written as

$$\hat{e}^0 \wedge \hat{e}^{b_{p+1}} \wedge \dots \wedge \hat{e}^{b_n} = e^{(p-n)\alpha\phi} e^{\beta\phi} (V + dx^0) \wedge e^{b_{p+1}} \wedge \dots \wedge e^{b_n} .$$

Combining these three elements, we deduce that the top line of (3.27) can be written as

$$\epsilon (-)^p e^{(2p-n)\alpha\phi} e^{\beta\phi} (V + dx^0) \wedge \star (u_p - V \wedge u_{p-1}) . \quad (3.28)$$

We now consider the bottom line of (3.27). Following a similar procedure to the top line, we compute the components of (\hat{u}_p) in terms of the n -dimensional vielbein basis:

$$\begin{aligned} (\hat{u}_p)_{0a_1 \dots a_{p-1}} &= \hat{e}_0^{\hat{\mu}_1} \hat{e}_{a_1}^{\hat{\mu}_2} \dots \hat{e}_{a_{p-1}}^{\hat{\mu}_p} (\hat{u}_p)_{\hat{\mu}_1 \dots \hat{\mu}_p} \\ &= e^{(p-1)\alpha\phi} e^{-\beta\phi} (u_{p-1})_{a_1 \dots a_{p-1}} , \end{aligned}$$

where we used the fact that $\hat{e}_0^{\mu_1} = 0$. The epsilon symbols are related by

$$\hat{\epsilon}^{0a_1 \dots a_{p-1}}_{b_p \dots b_n} = -\epsilon \hat{\epsilon}_0^{a_1 \dots a_{p-1}}_{b_p \dots b_n} = \epsilon^{a_1 \dots a_{p-1}}_{b_p \dots b_n} ,$$

and the vielbein basis decomposes as

$$\hat{e}^{b_p} \wedge \dots \wedge \hat{e}^{b_n} = e^{(p-1-n)\alpha\phi} e^{b_p} \wedge \dots \wedge e^{b_n} .$$

Putting everything together, the bottom line of (3.27) can be written as

$$e^{(2p-n)\alpha\phi} e^{-(\beta+2\alpha)\phi\star} u_{p-1} . \quad (3.29)$$

We can now substitute the expressions (3.28) and (3.29) into (3.27) to obtain a formula for the decomposition of the Hodge-star of a p -form

$$\hat{\star}\hat{u}_p = \epsilon e^{(2p-n)\alpha\phi} \left[e^{-(\beta+2\alpha)\phi} \epsilon \star u_{p-1} + (-)^p e^{\beta\phi} (V + dx^0) \wedge \star (u_p - V \wedge u_{p-1}) \right] .$$

Taking the wedge product between an arbitrary p -form \hat{v}_p and the Hodge star of an arbitrary p -form $\hat{\star}\hat{u}_p$ we find

$$\begin{aligned} \hat{v}_p \wedge \hat{\star}\hat{u}_p &= \epsilon e^{(2p-n)\alpha\phi} \left[v_p + dx^0 \wedge v_{p-1} \right] \\ &\quad \wedge \left[\epsilon e^{-(\beta+2\alpha)\phi} \star u_{p-1} + (-)^p e^{\beta\phi} (V + dx^0) \wedge \star (u_p - V \wedge u_{p-1}) \right] . \end{aligned}$$

Only the terms which contain precisely one dx^0 will survive. Using this fact and expanding out the brackets we find

$$\begin{aligned} \hat{v}_p \wedge \hat{\star}\hat{u}_p &= \epsilon e^{(2p-n)\alpha\phi} dx^0 \wedge \left[\epsilon e^{-(\beta+2\alpha)\phi} v_{p-1} \wedge \star u_{p-1} \right. \\ &\quad \left. + e^{\beta\phi} (v_p - V \wedge v_{p-1}) \wedge \star (u_p - V \wedge u_{p-1}) \right] . \end{aligned}$$

We now apply this decomposition to the \hat{S}_{gauge} term in our Lagrangian. We then perform the integration over the x^0 direction, normalising the circumference to be $\int dx^0 = 1$. We are left with the dimensionally reduced matter term

$$\begin{aligned} S_{\text{gauge}} &= (-)^f \int e^{(2p-n)\alpha\phi} \left[- e^{-(\beta+2\alpha)\phi} \frac{1}{2} F_{p-1} \wedge \star F_{p-1} \right. \\ &\quad \left. - \epsilon e^{\beta\phi} \frac{1}{2} (F_p - V \wedge F_{p-1}) \wedge \star (F_p - V \wedge F_{p-1}) \right] . \end{aligned}$$

This gives (3.21) when evaluating in terms of components and making the choice $p = 2$.

3.3.5 Reduction of $F \wedge F$ term

If the original number of dimensions $(n+1)$ is even and $(p+1) = \left(\frac{n+1}{2}\right)$ is also even, the action may admit a topological term of the form

$$\hat{S}_{\text{top}} = \int \frac{1}{2} d\hat{A}_p \wedge d\hat{A}_p .$$

We can decompose this $(p+1)$ -form into n -dimensional objects

$$d\hat{A}_p = dA_p + dx^0 \wedge dA_{p-1} .$$

We can now substitute this back into the action

$$\begin{aligned} \hat{S}_{\text{top}} &= \int \frac{1}{2} (dA_p + dx^0 \wedge dA_{p-1}) \wedge (dA_p + dx^0 \wedge dA_{p-1}) \\ &= \int dx^0 \wedge dA_p \wedge dA_{p-1} , \end{aligned}$$

where we used the fact that the action must contain precisely one dx^0 . Integrating out the compact dimension we are left with

$$S_{\text{top}} = \int dA_p \wedge dA_{p-1} ,$$

which gives (3.23) when expressed in components and choosing $p = 1$.

Chapter 4

The r-map

In this chapter we perform the dimensional reduction of a class of five-dimensional theories of gravity coupled to scalar and vector fields over a timelike or spacelike dimension. The class of theories under consideration generalises that of $5d, \mathcal{N} = 2$ vector multiplets coupled to supergravity in the sense that the scalar target manifold is only required to be a generalised projective special real manifold, as defined in chapter 2. Since the reduction of the supersymmetric theory goes by the name of the r-map, we call the reduction of the generalised theory the ‘generalised r-map’. We then discuss the geometry of the target manifold after dimensional reduction, and prove that for the supersymmetric theory this is a projective special Kähler or projective special para-Kähler manifold, depending on whether the reduction is over a spacelike or timelike dimension. None of the results presented in this chapter are original work of the author, and are based on the existing publications [27, 32]. However, the results are integral to the author’s works [60, 59], which will be presented in the next chapter. A discussion of the r-map also serves to complement our later chapter on the c-map (chapter 6), which uses techniques that are similar in spirit to those presented here.

We will discuss generalisations of $5d, \mathcal{N} = 2$ supergravity coupled to vector multiplets in section 4.1, before dimensionally reducing this class of theories in 4.2. We then prove the projective special Kähler and para-Kähler properties of spacelike and timelike reduction in sections 4.3 and 4.4.

4.1 Generalisations of $5D, \mathcal{N} = 2$ supergravity

Our starting point is the Lagrangian of a five-dimensional theory of gravity coupled to n scalar fields and $(n + 1)$ abelian vector fields that takes the form

$$\mathbf{e}_5^{-1} \mathcal{L}_5 = \frac{1}{2} R_5 - \frac{3}{4} \bar{a}_{xy}(\phi) \partial_{\hat{\mu}} \phi^x \partial^{\hat{\mu}} \phi^y - \frac{1}{4} a_{IJ}(\phi) \hat{F}_{\hat{\mu}\hat{\nu}}^I \hat{F}^{J\hat{\mu}\hat{\nu}} + \dots, \quad (4.1)$$

where spacetime indices run from $\hat{\mu} = 0, \dots, 3$, and internal indices run from $x = 1, \dots, n$ and $I = 0, \dots, n$. This Lagrangian is clearly of the same form as the bosonic part of $5d, \mathcal{N} = 2$ vector multiplets coupled to supergravity (3.3). However, we shall only impose weaker restrictions on the coupling matrices than those required by supersymmetry, and by doing so obtain a direct generalisation of the supersymmetric theory. The dots represent the fact that we leave open the option of including a Chern–Simons term such as that which appears in the supergravity Lagrangian. We do this for two reasons: first, it is not clear at present how we should deform this term away from the supergravity limit, and second it vanishes for the field configurations we will consider in

this thesis anyway. We may also include terms with fermionic fields in the Lagrangian, but these will never play a role in constructing bosonic background solutions.

We require that the non-linear sigma model takes values in an n -dimensional generalised projective special real target manifold, as defined in chapter 2. We denote by H the unique Hesse potential of the corresponding \mathfrak{d} -conic Hessian manifold that is homogeneous of degree \mathfrak{d} . This Hesse potential completely determines the dynamics of the Lagrangian. The scalar fields ϕ^x are required to parametrise the hypersurface $H(h^0, \dots, h^n) = 1$, and we shall use the particularly convenient parametrisation given by (2.14)

$$\phi^x = \frac{h^x}{h^0}, \quad h^0 = H\left(1, \frac{h^1}{h^0}, \dots, \frac{h^n}{h^0}\right)^{-\frac{1}{\mathfrak{d}}}, \quad (4.2)$$

where h^I are special coordinates in the \mathfrak{d} -conic Hessian manifold. The matrix a_{IJ} is defined according to (2.13)

$$a_{IJ} = -\frac{1}{\mathfrak{d}} \frac{\partial^2}{\partial h^I \partial h^J} \log H(h), \quad (4.3)$$

and can be written in terms of ϕ^x using (4.2). For the special case where the Hesse potential is a cubic polynomial this Lagrangian (plus Chern–Simons term) represents the bosonic part of the Lagrangian of $\mathcal{N} = 2$ supergravity coupled to n vector multiplets, see section 3.1.2.

We prefer not to work with the coordinates ϕ^x , which parametrise the physical hypersurface $H = 1$, but rather the special coordinates h^I , which live in the larger ambient space. To describe the same physics, it is understood that at some point we must restrict these coordinates to the hypersurface by imposing the constraint

$$H(h^0, \dots, h^n) = 1. \quad (4.4)$$

In terms of the Lagrangian this means that we should make the replacement

$$\bar{a}_{xy}(\phi) \partial_{\hat{\mu}} \phi^x \partial^{\hat{\mu}} \phi^y \longrightarrow a_{IJ}(h) \partial_{\hat{\mu}} h^I \partial^{\hat{\mu}} h^J,$$

while imposing the hypersurface constraint (4.4). Since the constraint fixes a symmetry of the Lagrangian it is acceptable to impose it before or after calculating the equations of motion. The full Lagrangian can then be written as

$$\mathbf{e}_5^{-1} \mathcal{L}_5 = \frac{1}{2} R_5 - \frac{3}{4} a_{IJ}(h) \partial_{\hat{\mu}} h^I \partial^{\hat{\mu}} h^J - \frac{1}{4} a_{IJ}(h) \hat{F}_{\hat{\mu}\hat{\nu}}^I \hat{F}^{J\hat{\mu}\hat{\nu}} + \dots, \quad (4.5)$$

where it is understood that the scalar fields are subject to the constraint (4.4). The couplings for the scalar fields and vector fields are now identical, and are given directly in terms of h^I through (4.3).

4.2 Dimensional reduction

We will now reduce the Lagrangian (4.5) over a timelike or spacelike dimension, which we keep track of by setting

$$\epsilon = \begin{cases} +1 & \text{timelike,} \\ -1 & \text{spacelike.} \end{cases}$$

Before we perform the calculation we shall impose two conditions that will greatly simplify the procedure, and refer the reader to [27] for a full account of the reduction

without making these assumptions. First, we shall impose that when we decompose the gauge vectors according to the manner adapted to dimensional reduction (see (3.22))

$$\hat{A}^I = \sqrt{3/2} b^I dx^0 + C_\mu^I dx^\mu, \quad (4.6)$$

the coefficients C_μ^I are constant. If the compact dimension is timelike then this corresponds to selecting backgrounds which are purely electric. This is natural if one wishes to consider black hole solutions, which in five dimensions carry only electric charge. After imposing this condition any Chern–Simons term will vanish identically, and the remaining field strength term simplifies according to

$$\hat{F}_{\hat{\mu}\hat{\nu}}^I \hat{F}^{J\hat{\mu}\hat{\nu}} = -\epsilon 3e^{-2\tilde{\sigma}} \partial_{\hat{\mu}} b^I \partial^{\hat{\mu}} b^J.$$

The second condition we will impose is that the isometry over which we dimensionally reduce defines an integrable distribution. This can be phrased in three alternative ways: (i) the compact dimension is orthogonal to a set of hypersurfaces that foliate spacetime, (ii) the Kaluza–Klein vector vanishes, or (iii) the metric is static (for a timelike distribution only). In any case, it means that we can always find coordinates in which the metric has no mixed terms

$$ds_5^2 = -\epsilon e^{2\tilde{\sigma}} (dx^0)^2 + e^{-\tilde{\sigma}} g_{\mu\nu} dx^\mu dx^\nu, \quad (4.7)$$

where we have decomposed the metric in accordance with the dimensional reduction procedure (see (3.20)).

We now perform the dimensional reduction over the x^0 dimension following the prescription outlined in section 3.3. The resulting four-dimensional Lagrangian is given by

$$\mathbf{e}_4^{-1} \mathcal{L}_4 = \frac{1}{2} R_4 - \frac{3}{4} \partial_\mu \tilde{\sigma} \partial^\mu \tilde{\sigma} - \frac{3}{4} a_{IJ}(h) (\partial_\mu h^I \partial^\mu h^J - \epsilon e^{-2\tilde{\sigma}} \partial_\mu b^I \partial^\mu b^J).$$

Recall that the scalars h^I are still required to satisfy the hypersurface constraint (4.4). It turns out that we can make a particular field redefinition that not only simplifies the Lagrangian but also eliminates the need for any constraints. We can do this by introducing new scalar fields σ^I , which we define by

$$\sigma^I := e^{\tilde{\sigma}} h^I. \quad (4.8)$$

The equation that constrained the h^I coordinates now reads

$$H(\sigma^0, \dots, \sigma^n) = e^{\mathbf{d}\tilde{\sigma}}, \quad (4.9)$$

which just gives an expression for the KK-scalar $\tilde{\sigma}$ in terms of the new coordinates. We can interpret this as follows: the KK-scalar has reinstated the direction orthogonal to H which was fixed by the hypersurface constraint, and so the new fields parametrise the whole \mathbf{d} -conic Hessian manifold. They are therefore not constrained to any particular hypersurface.

We would like to write the four-dimensional Lagrangian in terms of the new scalar fields. We first note that by differentiating the identity $a_{IJ}(h) h^I h^J = 1$ we find

$$a_{IJ}(h) h^I \partial_{\hat{\mu}} h^J = 0,$$

where we made use of the fact that the metric components $a_{IJ}(h)$ are homogeneous functions of degree -2 . It follows that

$$a_{IJ}(h) \partial_\mu h^I \partial^\mu h^J = a_{IJ}(\sigma) \partial_\mu \sigma^I \partial^\mu \sigma^J - \partial_\mu \tilde{\sigma} \partial^\mu \tilde{\sigma}.$$

The Lagrangian is therefore written in the new coordinates in a particularly neat form

$$\mathbf{e}_4^{-1} \mathcal{L}_4 = \frac{1}{2} R_4 - \frac{3}{4} a_{IJ}(\sigma) (\partial_\mu \sigma^I \partial^\mu \sigma^J - \epsilon \partial_\mu b^I \partial^\mu b^J) , \quad (4.10)$$

where all fields are unconstrained. In the next chapter we will use this formulation of the Lagrangian to calculate new non-extremal black hole solutions.

The underlying four-dimensional spacetime has Lorentzian signature if $\epsilon = -1$, and Euclidean signature if $\epsilon = 1$. The scalar fields σ^I, b^I describe a non-linear sigma model which maps four-dimensional spacetime into a $2(n+1)$ -dimensional target manifold with metric $\frac{3}{4} [a_{IJ} \oplus (-\epsilon a_{IJ})]$, where $a_{IJ}(\sigma)$ is a real, positive definite matrix that only depends on half the coordinates. In the next two sections we will prove that in the supergravity case (Hesse potential a cubic polynomial) the target manifold is projective special Kähler if $\epsilon = -1$, and projective special para-Kähler if $\epsilon = 1$.

4.3 Projective special Kähler structure of spacelike reduction

For spacelike reduction we make the choice $\epsilon = -1$. Using the expression (4.10) we can write the Lagrangian as

$$\mathbf{e}_4^{-1} \mathcal{L}_4 = \frac{1}{2} R_4 - \frac{3}{4} a_{IJ}(\sigma) (\partial_\mu \sigma^I \partial^\mu \sigma^J + \partial_\mu b^I \partial^\mu b^J) . \quad (4.11)$$

The scalar fields describe a non-linear sigma model from four-dimensional spacetime with Minkowski signature into a $2(n+1)$ -dimensional target manifold with metric $g = g_{ab} dq^a \otimes dq^b$, where we have gathered together the $2(n+1)$ coordinates into $q^a = (\sigma^I, b^I)^T$. The components of the metric are given by

$$g_{ab} = \frac{3}{4} \begin{pmatrix} a_{IJ} & 0 \\ 0 & a_{IJ} \end{pmatrix} .$$

An almost complex structure is defined by simply taking the canonical almost complex structure in any coordinate basis

$$J^a_b = \begin{pmatrix} 0 & \mathbb{1}_{(n+1)} \\ -\mathbb{1}_{(n+1)} & 0 \end{pmatrix} ,$$

with respect to which the metric is clearly Hermitian. A basis of the complexified co-tangent bundle $T^* M^{\mathbb{C}}$ is then given by

$$dX^I = d\sigma^I + i db^I , \quad d\bar{X}^I = d\sigma^I - i db^I ,$$

where dX^I and $d\bar{X}^I$ are bases of $T^{*1,0} M^{\mathbb{C}}$ and $T^{*0,1} M^{\mathbb{C}}$ respectively. Since these one-forms are exact the complex structure is integrable by virtue of the Newlander–Nirenberg theorem. The fundamental two-form is closed

$$d\omega = \frac{3}{4} d(a_{IJ} d\sigma^I \wedge db^J) = \frac{3}{4} (\partial_{I,J,K}^3 a) d\sigma^K \wedge d\sigma^I \wedge db^J = 0 ,$$

and, hence, the target manifold is Kähler.

We will now show that for the supergravity r-map, i.e. when the Hesse potential is a cubic polynomial, the target manifold has the property of being projective special

Kähler. We will follow the argument given in [27]. The idea is to show that projective special Kähler manifolds with prepotentials of the form

$$F(X) = \frac{1}{6} \frac{(CXXX)}{X^0} \quad (4.12)$$

directly correspond to those obtained from the r-map. Here we have introduced the notation $(CXXX) = C_{ABC}X^AX^BX^C$ and $(CXX)_A = C_{ABC}X^BX^C$ etc. First, the derivatives of the prepotential with respect to X^I are given by

$$F_0 = -\frac{1}{6} \frac{(CXXX)}{(X^0)^2}, \quad F_A = \frac{1}{2} \frac{(CXX)_A}{X^0}.$$

This implies that \mathcal{F} , as defined in section 3.1.3, and $z^A = X^A/X^0$ satisfy the expressions

$$\mathcal{F} = \frac{1}{6}(Czzz), \quad \mathcal{F}_A = \frac{1}{2}(Czz)_A.$$

The expression for $e^{-\bar{K}}$, as defined in section 3.1.3, is given by

$$\begin{aligned} e^{-\bar{K}} &= -i \left[\frac{1}{6} \left((Czzz) - (C\bar{z}\bar{z}\bar{z}) \right) - \frac{1}{2} (z^A - \bar{z}^A) \left((Czz)_A + (C\bar{z}\bar{z})_A \right) \right] \\ &= -\frac{i}{6} C_{ABC} (z^A - \bar{z}^A) (z^B - \bar{z}^B) (z^C - \bar{z}^C) = -\frac{8}{6} (Cyyy), \end{aligned}$$

where $y^A := \text{Im}(z^A)$. The metric is then given through equation (3.8) by

$$g = \left(-\frac{3}{2} \frac{(Cy)_{AB}}{(Cyyy)} + \frac{9}{4} \frac{(Cyy)_A (Cyy)_B}{(Cyyy)^2} \right) (dx^A \otimes dx^B + dy^A \otimes dy^B),$$

where $x^A := \text{Re}(z^A)$. Compare this with the metric of the target manifold of the sigma-model appearing in (4.11):

$$g = \left(-\frac{3}{2} \frac{(C\sigma)_{AB}}{(C\sigma\sigma\sigma)} + \frac{9}{4} \frac{(C\sigma\sigma)_A (C\sigma\sigma)_B}{(C\sigma\sigma\sigma)^2} \right) (d\sigma^A \otimes d\sigma^B + db^A \otimes db^B).$$

It is then clear that, by making the identification $\sigma^A \leftrightarrow y^A$ and $b^A \leftrightarrow x^A$, projective special Kähler manifolds with prepotentials of the form (4.12) give precisely the same metrics as those obtained through the r-map. This completes the proof that the target manifold in the image of the supergravity r-map is projective special Kähler.

Let us end with two remarks. First, one may interpret the supergravity r-map geometrically as a map from a projective special real manifold into the tangent bundle. For more details we refer the reader to [34, 9] and references therein. Second, one might also wonder what the target manifold for the generalised r-map is, and not just in the supersymmetric case. This is currently under investigation [28].

4.4 Projective special para-Kähler structure of timelike reduction

For timelike reduction we make the choice $\epsilon = +1$. Using expression (4.10) we can write the Lagrangian as

$$\mathbf{e}_4^{-1} \mathcal{L}_4 = \frac{1}{2} R_4 - \frac{3}{4} a_{IJ}(\sigma) (\partial_\mu \sigma^I \partial^\mu \sigma^J - \partial_\mu b^I \partial^\mu b^J). \quad (4.13)$$

The scalar part of the Lagrangian describes a non-linear sigma model into a $2(n+1)$ -dimensional target manifold with metric $g = g_{ab} dq^a \otimes q^b$, where again we have gathered together the $2(n+1)$ coordinates into $q^a = (\sigma^I, b^I)^T$. This time an additional minus sign appears in the components of the metric

$$g_{ab} = \frac{3}{2} \begin{pmatrix} a_{IJ} & 0 \\ 0 & -a_{IJ} \end{pmatrix} ,$$

which reflects the fact that the target manifold has neutral signature. We define an almost para-complex structure by taking the canonical para-complex structure in any coordinate basis

$$J^a_b = \begin{pmatrix} 0 & \mathbb{1}_{(n+1)} \\ \mathbb{1}_{(n+1)} & 0 \end{pmatrix} .$$

The metric is clearly para-Hermitian with respect to this para-complex structure. A basis of the co-tangent bundle T^*M is given by

$$dX^I = d\sigma^I + db^I , \quad d\bar{X}^I = d\sigma^I - db^I ,$$

where dX^I and $d\bar{X}^I$ are bases of $T^{*+}M$ and $T^{*-}M$ respectively. Since these one-forms are exact the para-complex structure is integrable by Frobenius' theorem. The fundamental two-form is closed

$$d\omega = \frac{3}{2} d(a_{IJ} d\sigma^I \wedge db^J) = \frac{3}{2} (\partial_{I,J,K}^3 a) d\sigma^K \wedge d\sigma^I \wedge db^J = 0 ,$$

and, hence, the target manifold is para-Kähler.

We will show that for the temporal version of the supergravity r-map the target manifold has the property of being projective special para-Kähler. We follow again the argument presented in [27], where the idea is to show that the para-holomorphic prepotentials of the form

$$F(X) = \frac{1}{6} \frac{(CXXX)}{X^0} \tag{4.14}$$

give rise to precisely those metrics which appear as target manifolds of (4.13). By an identical calculation to the previous section, one can show that the metric corresponding to the prepotential (4.14) is given by

$$g = \left(-\frac{3}{2} \frac{(Cy)_{AB}}{(Cyyy)} + \frac{9}{4} \frac{(Cyy)_A (Cyy)_B}{(Cyyy)^2} \right) (dx^A \otimes dx^B + dy^A \otimes dy^B) ,$$

where $y^A := \text{Im}(z^A)$ and $x^A := \text{Re}(z^A)$. Compare this with the target manifold of the reduced Lagrangian (4.13)

$$g = \left(-\frac{3}{2} \frac{(C\sigma)_{AB}}{(C\sigma\sigma\sigma)} + \frac{9}{4} \frac{(C\sigma\sigma)_A (C\sigma\sigma)_B}{(C\sigma\sigma\sigma)^2} \right) (d\sigma^A \otimes d\sigma^B + db^A \otimes db^B) .$$

It is clear by making the identification $\sigma^A \leftrightarrow y^A$ and $b^A \leftrightarrow x^A$ that both metrics agree. We conclude that the target manifold in the image of the temporal version of the supergravity r-map is projective special para-Kähler.

Chapter 5

Five-dimensional black holes

In this chapter we will construct new black hole solutions to the class of theories introduced in section 4.1, which generalises that of $5d$, $\mathcal{N} = 2$ supergravity coupled to vector multiplets. Since we will only consider non-rotating black hole solutions we may impose that backgrounds are static and purely electric. In this case we can take the Lagrangian (4.13) as our starting point

$$\mathbf{e}_4^{-1} \mathcal{L}_4 = \frac{1}{2} R_4 - \frac{3}{4} a_{IJ}(\sigma) (\partial_\mu \sigma^I \partial^\mu \sigma^J - \partial_\mu b^I \partial^\mu b^J) . \quad (5.1)$$

It is worth emphasising that this is an effective four-dimensional Lagrangian for static and electric solutions of the full five-dimensional theory. This means that any solution of this Lagrangian corresponds to a static and electric solution of the original five-dimensional theory and vica-versa. Since not all static and electric solutions will correspond to black holes we must identify those which exhibit an event horizon and asymptote to Minkowski space.

Some of the simplest types of solutions one may consider are extremal instantons. We define these to be solutions that have a flat four-dimensional metric, and their name comes from the fact that they lift to extremal black holes in five dimensions. Such solutions can be found for generic models. A subclass of these solutions will correspond to BPS black holes, as the BPS condition is stronger than the extremality condition. This discussion on extremal solutions is not based on the author's work, but appears in [32]. We then go on to investigate non-extremal instanton solutions, by deforming four-dimensional spacetime away from the flat limit. This discussion is based on original works by the author [60, 59]. Since we construct non-extremal solutions in a systematic way, we have a good understanding of their properties. One interesting and novel feature that emerges is that in order for these solutions to lift to non-extremal black holes in five dimensions with smooth scalar geometries the number of independent integration constants must halve. This is similar to the extremal case, and suggests that a first order rewriting of the solution is always possible, when in general the equations of motion are second order. We also find a set of equations that describe the near horizon geometry of the scalar fields, which directly generalise the attractor equations for the extremal solution.

In section 5.1 we derive the equations of motion corresponding to the Lagrangian (5.1). We then introduce a natural set of dual coordinates in which the equations of motion take a particularly simple form. We then discuss extremal instanton solutions (i.e. a flat four-dimensional metric) in 5.2. These solutions are shown to satisfy first order equations and lift to extremal black holes in five dimensions. In section 5.3 we

construct new non-extremal black hole solutions. We first consider *STU*-like models, where we can integrate the second order equations of motion directly, and the general solution can be found. We then investigate the criteria for solutions to lift to non-extremal black holes in five dimensions. Finally, we discuss non-extremal solutions to generic models, in which case we find solutions with as many independent scalar fields as we have blocks in the metric. Since all metrics have at least one block this provides a universal solution with constant scalar fields for all models, which is none other than the Reissner–Nordström solution.

5.1 Equations of motion and dual coordinates

By considering the variation of (5.1) with respect to σ^I, b^I and $g_{\mu\nu}$, the equations of motion are found to be

$$\begin{aligned} \nabla^\mu [a_{IJ}\partial_\mu\sigma^J] - \frac{1}{2}\partial_I a_{JK} (\partial_\mu\sigma^J\partial^\mu\sigma^K - \partial_\mu b^J\partial^\mu b^K) &= 0, \\ \nabla^\mu [a_{IJ}\partial_\mu b^J] &= 0, \\ \frac{3}{4}a_{IJ} (\partial_\mu\sigma^I\partial_\nu\sigma^J - \partial_\mu b^I\partial_\nu b^J) &= \frac{1}{2}R_{\mu\nu}. \end{aligned}$$

The fact that the target manifold of the original theory is Hessian allows us to introduce a natural set of dual coordinates σ_I , defined by

$$\sigma_I := \frac{\partial}{\partial\sigma^I}\tilde{H}(\sigma). \quad (5.2)$$

For example in the *STU* model, where $H = \sigma^1\sigma^2\sigma^3$, and therefore $\tilde{H} = -\frac{1}{3}\log\sigma^1\sigma^2\sigma^3$, the dual coordinates are given by $\sigma_I = -\frac{1}{3}(\sigma^I)^{-1}$. For supersymmetric theories (i.e. when the Hesse potential is a cubic polynomial) the dual coordinates are always related algebraically to the original coordinates. Note that by the chain rule we have

$$\partial_\mu\sigma_I = \partial_{I,J}^2\tilde{H}(\sigma)\partial_\mu\sigma^J = a_{IJ}\partial_\mu\sigma^J. \quad (5.3)$$

For this reason it is often convenient to think of the metric a_{IJ} as an operator which raises and lowers indices; however one must be careful to note that an additional minus sign is acquired when raising and lower the index of the coordinate compared to its derivative

$$a_{IJ}\sigma^J = -\sigma_I, \quad a_{IJ}\partial_\mu\sigma^J = \partial_\mu\sigma_I.$$

At the same time as introducing the dual coordinates for the σ^I fields, it is convenient to define

$$\partial_\mu b_I = a_{IJ}\partial_\mu b^J. \quad (5.4)$$

This places an integrability condition on the solutions to the b^I fields, which will automatically be solved for all solutions presented in this thesis. Using the expressions (5.3) and (5.4) we can write the remaining equations of motion as

$$\Delta\sigma_I + \frac{1}{2}\partial_I a^{JK} (\partial_\mu\sigma_J\partial^\mu\sigma_K - \partial_\mu b_J\partial^\mu b_K) = 0, \quad (5.5)$$

$$\Delta b_I = 0, \quad (5.6)$$

$$\frac{3}{4}a^{IJ} (\partial_\mu\sigma_I\partial_\nu\sigma_J - \partial_\mu b_I\partial_\nu b_J) = \frac{1}{2}R_{\mu\nu}. \quad (5.7)$$

5.2 Extremal black holes

Recall that one particular feature of the four-dimensional extremal Reissner–Nordström black hole is the fact that the three-dimensional part of the metric is flat (3.14). This is therefore a natural metric ansatz to use when constructing extremal solutions [48]. In this case the Einstein equations (5.7) reduce to

$$\frac{1}{4}a^{IJ}(\partial_\mu\sigma_I\partial_\nu\sigma_J - \partial_\mu b_I\partial_\nu b_J) = 0 . \quad (5.8)$$

These can clearly be solved by setting

$$\partial_\mu\sigma_I = \pm\partial_\mu b_I , \quad (5.9)$$

where the choice of sign is the same for all I . Upon making this ansatz the second order equations of motion for both the σ^I and b^I fields reduce to

$$\Delta\sigma_I = 0 .$$

Solutions can therefore be characterised in terms of the harmonic functions

$$\mathcal{H}_I := A_I + \sum_\alpha \frac{Q_{\alpha I}}{|x - x_\alpha|^2} , \quad (5.10)$$

by setting $\sigma_I = \pm\mathcal{H}_I$ and $\partial_\mu b_I = \partial_\mu\mathcal{H}_I$. Clearly the choice of sign corresponds to the choice of sign in (5.9).

The solution given by (5.9) does not exhaust all the possibilities. A field rotation matrix $R^I{}_J$ is defined to be a constant matrix that satisfies

$$a_{IJ}R^I{}_K R^J{}_L = a_{KL} .$$

Clearly any field rotation matrix must be invertible, and we say it is trivial if $R = \pm\mathbb{1}$. For example, since the metric of the STU model is diagonal a non-trivial field rotation matrix is given by

$$R = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} , \quad (5.11)$$

where not all signs are equal. Geometrically the ansatz (5.9) corresponds to selecting a totally isotropic submanifold of the target space, and a field rotation matrix corresponds to a discrete isometry of this submanifold that is of the form

$$\sigma^I \rightarrow \sigma^I , \quad b^I \rightarrow R^I{}_J b^J .$$

The existence of non-trivial field rotation matrices depends on the particular model in question, and they are not guaranteed to exist generically. However, when a non-trivial field rotation matrix does exist we can generalise the solution (5.9) to

$$\partial_\mu\sigma_I = R_I{}^J \partial_\mu b_J , \quad (5.12)$$

where $R_I{}^J$ is the inverse transpose of $R^I{}_J$. These solutions can be written in terms of the harmonic functions (5.10) by setting $\sigma_I = R_I{}^J \mathcal{H}_J$ and $\partial_\mu b_I = \partial_\mu\mathcal{H}_I$. Now we see why the field rotation matrix gets its name: it rotates the charges of the solution relative to the scalar fields. When lifting solutions to black holes in five dimensions,

switching on a non-trivial field rotation matrix deforms a BPS solution into a non-BPS solution [50, 31].

In order for these solutions (either with or without a field rotation matrix) to lift to black holes in five dimensions we need to ensure that the solutions are asymptotically Minkowski, i.e. $e^{\tilde{\sigma}} \rightarrow 1$ as $r \rightarrow \infty$. This places one constraint on the integration constants A_I . We are therefore left with a solution defined by n independent constants of integration (the A_I plus one constraint). This is half what one would expect from the general solution to n second order differential equations obtained from the n physical scalar fields. It suggests that extremal black hole solutions actually satisfy first order differential equations (this is, of course, true since they are solutions to (5.9) or (5.12)). For the BPS case this can be understood by the fact that they solve Killing spinor equations, which are first order equations. Much has been discussed in the literature about the relation between extremal black holes and first order equations, e.g. [56], and it appears to be a common feature of all extremal solutions.

We will now show that the instanton solutions we have constructed to the four-dimensional Euclidean theory lift to extremal black holes in five dimensions. This amounts to retracing our steps in the dimensional reduction procedure. Firstly, the Kaluza–Klein scalar can be read off from equation (4.9)

$$e^{\tilde{\sigma}} = H(\sigma^0, \dots, \sigma^n)^{\frac{1}{d}}. \quad (5.13)$$

Of course, to obtain a truly explicit formula for $e^{\tilde{\sigma}}$ in terms of the solution $\sigma_I = \pm \mathcal{H}_I$ we would need to solve equation (5.2), which relates the coordinates to the dual coordinates and is model dependent. This amounts to solving what is referred to in the literature as the ‘generalised stabilisation equations’, which describe the scalar fields in terms of harmonic functions. Even for algebraic equations this may not be possible in closed form, and so if we want to write down explicit solutions we must rely on known models where these equations are solvable. We can substitute (5.13) into (4.7) to obtain an expression for the five-dimensional line element. As discussed, imposing $e^{\tilde{\sigma}} \rightarrow 1$ as $r \rightarrow \infty$ ensures that this line element is asymptotically flat, and imposes one constraint on the integration constants A_I . The area of the event horizon only depends on the charges, and we therefore observe attractor behaviour.

Now that we have an expression for the KK-scalar, we can find expressions for the h^I coordinates through (4.8). From these we obtain expressions for the physical scalar fields ϕ^x using (4.2). The solutions to the gauge fields are given in terms of harmonic functions through (5.4) and (4.6). This determines the solution of the five-dimensional fields completely. Note that the five-dimensional dual scalar fields h_I satisfy the expression

$$e^{-\tilde{\sigma}} h_I = \mathcal{H}_I, \quad (5.14)$$

which generalises the known form of the generalised stabilisation equations in five dimensions [45, 32].

Let us end with the example of the *STU* model, where we can write down a solution explicitly [32]. For concreteness we shall consider the spherically symmetric solution with just one centre

$$\mathcal{H}_I = A_I + \frac{Q_I}{r^2},$$

where we are using coordinates such that $r = 0$ corresponds to the horizon. The BPS

solution is given by

$$ds^2 = -(\mathcal{H}_0\mathcal{H}_1\mathcal{H}_2)^{-\frac{2}{3}}dt^2 + (\mathcal{H}_0\mathcal{H}_1\mathcal{H}_2)^{\frac{1}{3}}\delta_{\mu\nu}dx^\mu dx^\nu ,$$

$$h^I = \frac{(\mathcal{H}_0\mathcal{H}_1\mathcal{H}_2)^{\frac{1}{3}}}{\mathcal{H}_I} .$$

The physical scalar fields are given through (4.2) by

$$\phi^1 = \frac{\mathcal{H}_0}{\mathcal{H}_1} , \quad \phi^2 = \frac{\mathcal{H}_0}{\mathcal{H}_2} .$$

On the horizon they take the values that depend only on the charges and not the values at radial infinity, i.e. they exhibit attractor behaviour

$$\phi^1 \xrightarrow{r \rightarrow 0} \frac{Q_0}{Q_1} , \quad \phi^2 \xrightarrow{r \rightarrow 0} \frac{Q_0}{Q_2} .$$

This behaviour can be characterised by the expressions

$$Zh_I \xrightarrow{r \rightarrow 0} Q_I ,$$

where we have defined the five-dimensional central charge $Z := -h^I Q_I$. The above expression is the standard form of the attractor equations in five dimensions [44, 45]. In order for the solution to asymptote to Minkowski space, i.e. $e^{\tilde{\sigma}} \rightarrow 1$ as $r \rightarrow \infty$, we must impose the following constraint on the integration constants

$$A_0 A_1 A_2 = 1 .$$

We are left with a solution characterised by the electric charges and two independent integration constants, as we expected from our previous discussion. This solution is a direct generalisation of the five-dimensional extremal Reissner–Nordström black hole, which can be obtained in the limit where all three harmonic functions are proportional. One may obtain non-BPS solutions to this model by introducing a non-trivial field rotation matrix of the form (5.11).

5.3 Non-extremal black holes

In this section we present the non-extremal black hole solutions found in [60, 59] and discuss their properties. We will first present solutions of *STU*-like models, which is a class of models that includes the *STU* model and particular generalisations thereof. For this class of models we find the general solution with a full set of independent scalar fields. We then discuss solutions to generic models. By decomposing the target space metric a_{IJ} into block diagonal form (where possible) we obtain solutions with as many independent scalar fields as there are blocks in the metric. Since all metrics can be said to contain at least one block, this provides one universal solution with constant scalar fields to all models, which is the Reissner–Nordström solution.

Since we are interested in non-extremal solutions we will not impose that the four-dimensional metric is flat, but we will impose that all fields are spherically symmetric. This is, in fact, enough to completely fix the four-dimensional metric. This can be seen as follows: the four-dimensional part of a general spherically symmetric five-dimensional metric can be written as [49]

$$ds_4^2 = e^{6A(\tau)} d\tau^2 + e^{2A(\tau)} d\Omega_{(3)}^2 ,$$

where τ is an affine radial parameter, which means the Laplace operator takes the form

$$\Delta_5 = \frac{d^2}{d\tau^2} + \text{terms independent of } \tau .$$

Since we have imposed that solutions are spherically symmetric the LHS of the Einstein equations when either $\mu \neq \tau$ or $\nu \neq \tau$ must vanish. On the RHS of these equations there are terms proportional to $\ddot{A} - 2e^{4A}$, which implies that $\ddot{A} - 2e^{4A} = 0$. This can be integrated once to get $\dot{A}^2 = c^2 + e^{4A}$, where we choose c^2 to be positive so that we avoid solutions periodic in the radial coordinate. By relabelling $y = e^{-2A}$ this can be written as

$$\dot{y} = \pm 2\sqrt{c^2 y^2 + 1} ,$$

and the solution is given by

$$y(\tau) = \frac{\sinh(\pm 2c\tau + D)}{c} .$$

To ensure that y is strictly positive in the range $0 < \tau < +\infty$ we choose the positive sign and $D = 0$. We can then write the solution as

$$e^{-2A(\tau)} = y(\tau) = \frac{\sinh(2c\tau)}{c} .$$

The four-dimensional line element then takes the form

$$ds_4^2 = \frac{c^3}{\sinh^3(2c\tau)} d\tau^2 + \frac{c}{\sinh(2c\tau)} d\Omega_{(3)}^2 . \quad (5.15)$$

This is, in fact, nothing other than the four-dimensional part of the standard Reissner–Nordström solution¹. To see this we define the new radial coordinate r through the expression

$$r^2 = \frac{ce^{2c\tau}}{\sinh(2c\tau)} .$$

We can then write the four-dimensional line element as

$$ds_4^2 = W^{-\frac{1}{2}} dr^2 + W^{\frac{1}{2}} r^2 d\Omega_3^2 ,$$

where

$$W = 1 - \frac{2c}{r^2} = e^{-4c\tau} . \quad (5.16)$$

This is precisely the four-dimensional part of the five-dimensional non-extremal Reissner–Nordström solution [60], according to the decomposition (4.7). When constructing solutions it is often easier to work with the affine parameter τ rather than the standard radial coordinate. This is valid in the range $0 < \tau < +\infty$, where radial infinity is given by the limit $\tau \rightarrow 0^+$ and the outer horizon is given by $\tau \rightarrow +\infty$. The radial coordinate r is valid up to the inner horizon at $r = 0$, with the outer horizon located at $r^2 = 2c$. Harmonic functions with respect to the flat four-dimensional metric, which we shall always denote by \mathcal{H}_I , can be written in terms of τ as

$$\mathcal{H}_I := A_I + \frac{Q_I}{r^2} = \left(A_I + \frac{Q_I}{2c} \right) - \frac{Q_I}{2c} e^{-4c\tau} .$$

¹This fact was first noted in the reduction from four to three dimensions in [30], and one may refer to [49, 103] for the corresponding argument in arbitrary dimensions.

Let us return to the equations of motion. We have seen that after imposing spherical symmetry the four-dimensional line element reduces to (5.15), in which case the (τ, τ) component of the Ricci tensor is given by $R_{\tau\tau} = 6c^2$. The equations of motion can therefore be written as

$$\ddot{\sigma}_I + \frac{1}{2}\partial_I a^{JK}(\dot{\sigma}_J \dot{\sigma}_K - \dot{b}_I \dot{b}_K) = 0, \quad (5.17)$$

$$\ddot{b}_I = 0, \quad (5.18)$$

$$\frac{1}{4}a^{IJ}(\dot{\sigma}_I \dot{\sigma}_J - \dot{b}_I \dot{b}_J) = c^2. \quad (5.19)$$

The equations (5.17) and (5.18) follow from the effective one-dimensional action

$$\mathcal{L} = \frac{1}{4}a_{IJ}(\sigma) \left(\dot{\sigma}^I \dot{\sigma}^J - \dot{b}^I \dot{b}^J \right),$$

and must then be supplemented by (5.19), which can be interpreted as a Hamiltonian constraint. The equations (5.17) can be solved immediately to give

$$\dot{b}_I = Q_I,$$

where the Q_I are the constant electric charges carried by the solution.

We are left to find solutions to (5.17) and (5.18), which may lift to non-extremal black holes in five dimensions. We should recover our extremal solutions by taking the limit $c \rightarrow 0$. Unfortunately the presence of the non-extremality parameter c makes the task of finding solutions considerably harder than the extremal case, as it entangles the second order equations in a highly non-trivial manner. We will only be able to find a complete set of independent solutions to *STU*-like models. However, we can still find solutions to generic models where some scalar fields are proportional to each other.

5.3.1 *STU*-like models

We define an *STU*-like model to be a model with Hesse potential of the form

$$H(h^0, \dots, h^n) = \frac{1}{\mathbf{d}(\mathbf{d}-1)} (h^0 h^1 \dots h^n)^{\frac{\mathbf{d}}{(n+1)}},$$

or models that can be brought to this form by a linear transformation. We will only consider coordinate patches where the coordinates h^I are pointwise non-zero, and by definition the Hesse potential is strictly positive. The supergravity *STU* model is given by the special case $n = 2$, and $\mathbf{d} = 3$.

For this class of models the dual coordinates are given by

$$\sigma_I = -\frac{1}{(n+1)\sigma^I}.$$

The metric a_{IJ} is diagonal, and can be written in terms of the dual coordinates as

$$a_{IJ} = (n+1)\text{diag}(\sigma_0^2, \dots, \sigma_n^2).$$

The derivative of the inverse is given by

$$\partial_I a^{JK} = \text{diag}\left(-\frac{2}{\sigma_0}, \dots, -\frac{2}{\sigma_n}\right).$$

We can therefore write the equations of motion (5.17) and (5.19) as

$$\ddot{\sigma}_I - \frac{[(\dot{\sigma}_I)^2 - (Q_I)^2]}{\sigma_I} = 0, \quad (5.20)$$

$$\sum_I \frac{[(\dot{\sigma}_I)^2 - (Q_I)^2]}{(n+1)\sigma_I^2} = 4c^2. \quad (5.21)$$

Observe that for this class of models the second order equations of motion completely decouple from one another, and, in fact, can be integrated explicitly to give the general solution to this system of equations

$$\sigma_I = \pm \frac{Q_I}{B_I} \sinh \left(B_I \tau + B_I \frac{A_I}{Q_I} \right). \quad (5.22)$$

Substituting this solution into the constraint equation (5.21) gives us an expression for c in terms of the integration constants B_I :

$$\frac{1}{(n+1)}(B_0)^2 + \dots + \frac{1}{(n+1)}(B_n)^2 = 4c^2. \quad (5.23)$$

The solution (5.22) is invariant under the exchange $B_I \leftrightarrow -B_I$, and so without loss of generality we may assume that the B_I are non-negative. Taking the extremal limit $c \rightarrow 0$ at this stage forces the B_I to vanish due to (5.23), and so the general solution reduces to the spherically symmetric extremal solution.

We will now lift these solutions back up to five dimensions. The first task is to use equation (4.9) to determine the KK-scalar for this solution

$$e^{-\tilde{\sigma}} = (-1)^{(n+1)}(n+1)(\sigma_0\sigma_1\dots\sigma_n)^{\frac{d}{(n+1)}}.$$

This is all we need to obtain the five-dimensional line element through (4.7)

$$ds_5^2 = -\frac{1}{(n+1)^2(\sigma_0\dots\sigma_n)^{\frac{2}{(n+1)}}}dt^2 \\ + (-1)^{(n+1)}(n+1)(\sigma_0\dots\sigma_n)^{\frac{1}{(n+1)}} \left(\frac{c^3}{\sinh^3 2c\tau} d\tau^2 + \frac{c}{\sinh 2c\tau} d\Omega_{(3)}^2 \right).$$

We now need to determine whether or not these solutions correspond to black holes. Firstly, to ensure that the solutions are asymptotically Minkowski we must require that $e^{\tilde{\sigma}} \rightarrow 1$ as $\tau \rightarrow 0^+$. This places one constraint on the integration constants A_I . We then need to check that the area of the horizon is finite. This is calculated to be

$$A = \lim_{\tau \rightarrow +\infty} (-1)^{(n+1)}(n+1)(\sigma_0\dots\sigma_n)^{\frac{1}{(n+1)}} \frac{c}{\sinh 2c\tau}.$$

Since the B_I are non-negative the highest order term in the numerator is given by $\exp(\frac{1}{(n+1)}(B_0 + \dots + B_n)\tau)$. In order for the limit to converge this must cancel with the highest order term in the denominator, which is given by $\exp(2c\tau)$. We conclude that in order to have a finite area we must require

$$\frac{1}{(n+1)}(B_0 + \dots + B_n) = 2c. \quad (5.24)$$

We now focus on the physical scalar fields, which can be written in terms of the dual coordinates as

$$\phi^x = \frac{\sigma_0}{\sigma_x}.$$

In order for these fields to take finite values on the horizon the highest order term in the numerator, $\exp(B_0\tau)$, must cancel with the highest term in the denominator, $\exp(B_x\tau)$. If we combine this with the requirement of having a finite area (5.24), we conclude that the integration constants B_I must all be equal to $2c$

$$B_0 = \dots = B_n = 2c. \quad (5.25)$$

The solution (5.22) therefore reduces to

$$\sigma_I = \pm \frac{Q_I}{2c} \sinh \left(2c\tau + 2c \frac{A_I}{Q_I} \right). \quad (5.26)$$

This solution has n independent integration constants: the $(n+1)$ constants A^I subject to the constraint $e^{\tilde{\sigma}} \rightarrow 1$ at $\tau \rightarrow 0^+$. One would naively expect a solution to n second order equations of motion to have $2n$ independent integration constants. The fact that the number of constants drops by half suggests that we can rewrite the solution in terms of first order equations. This is certainly the case, as the solution (5.26) satisfies the first order equations

$$\dot{\sigma}_I = \pm \sqrt{Q_I^2 + 4c^2 \sigma_I^2}.$$

We can also write the solution in terms of gradient flow equations by identifying a generating function $\mathcal{W} = \mathcal{W}(\sigma^I, Q_I, c)$ that satisfies

$$\dot{\sigma}_I = \partial_I \mathcal{W}.$$

Such a generating function is given by

$$\mathcal{W} := \pm \frac{1}{(n+1)} \sum_I \left[\sqrt{4c^2 + (n+1)^2 Q_I^2 \sigma^{I^2}} + c \log \left(\frac{\sqrt{4c^2 + (n+1)^2 Q_I^2 \sigma^{I^2}} - 2c}{\sqrt{4c^2 + (n+1)^2 Q_I^2 \sigma^{I^2}} + 2c} \right) \right],$$

which takes a similar form as the *STU* model in the four-dimensional case [104]. In the extremal limit the generating function reduces simply to $\mathcal{W} = \pm \sigma^I Q_I$, which reproduces the result of [32].

Let us collect everything together and write down the final solution for *STU*-like models that describes a non-extremal black hole. The line element is given by

$$ds_5^2 = - \frac{W}{(\mathcal{H}_0 \dots \mathcal{H}_n)^{\frac{2}{(n+1)}}} dt^2 + (\mathcal{H}_0 \dots \mathcal{H}_n)^{\frac{1}{(n+1)}} \left(\frac{dr^2}{W} + r^2 d\Omega_{(3)}^2 \right),$$

where

$$W = 1 - \frac{2c}{r^2}, \quad \mathcal{H}_I = \mp (n+1) \left[\frac{Q_I}{2c} \sinh \left(2c \frac{A_I}{Q_I} \right) + \frac{Q_I e^{-2c \frac{A_I}{Q_I}}}{2} \frac{1}{r^2} \right],$$

and the scalar fields are given in terms of the radial coordinate r or τ by

$$\phi^x = \frac{\sigma_0}{\sigma_x}, \quad \sigma_I = \frac{-1}{(n+1)} \frac{\mathcal{H}_I}{\sqrt{W}} = \pm \frac{Q_I}{2c} \sinh \left(2c\tau + 2c \frac{A_I}{Q_I} \right).$$

The constant parameters A_I must satisfy the constraint

$$\lim_{\tau \rightarrow 0^+} e^{\tilde{\sigma}} = 1 .$$

For the case where $n = 2$ and $d = 3$ this reproduces the non-extremal black hole solutions of $5D, \mathcal{N} = 2$ supergravity originally found in [105, 106]. Taking the extremal limit $c \rightarrow 0$ one recovers the spherically symmetric extremal solution, and we find no free parameters other than c appearing in the non-extremal solution.

On the horizon the physical scalar fields no longer depend only on the charges, but acquire a dependence on the value of the moduli at infinity

$$\phi^x \xrightarrow{r \rightarrow 0} \frac{Q_0 e^{2c \frac{A_0}{Q_0}}}{Q_x e^{2c \frac{A_x}{Q_x}}} ,$$

and the generalised stabilisation equations now read

$$e^{-\tilde{\sigma}} h_I = \frac{\mathcal{H}_I}{\sqrt{W}} .$$

The solution satisfies the expression

$$Z h_I \xrightarrow{r \rightarrow 0} Q_I e^{2c \frac{A_I}{Q_I}} ,$$

where we have defined $Z = -h^I Q_I e^{2c \frac{A_I}{Q_I}}$. These equations can be said to generalise the attractor equations to include non-extremal solutions. Note that the term ‘attractor equations’ is now just a label – the non-extremal solutions no longer exhibit attractor behaviour. Taking the extremal limit $c \rightarrow 0$ we recover the known five-dimensional attractor equations [44, 45]

5.3.2 Block diagonal models

We may classify any metric according to the number of blocks that appear when it is written as a matrix, e.g. the following metric has two blocks:

$$a_{IJ} = \begin{pmatrix} * & \dots & * & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & 0 & 0 & 0 \\ * & \dots & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & \dots & * \\ 0 & 0 & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & * & \dots & * \end{pmatrix} .$$

One may, of course, perform row operations to bring a metric into a form where it has as small blocks as possible. Such row operations correspond to taking general linear combinations of the scalar fields. Generically a metric will only contain one block, but for various classes of models it will split into more than one block. For example a Hesse potential of the form

$$H(\sigma^0, \dots, \sigma^n) = H^1(\sigma^0, \dots, \sigma^{k-1}) H^2(\sigma^k, \dots, \sigma^n) ,$$

has a corresponding metric with at least two blocks: one of size $k \times k$, and a second of size $l \times l$ where $l = (n + 1 - k)$. *STU*-like models have diagonal metrics, which have the maximum number of blocks possible.

For any model we can find solutions, based on the solutions to *STU*-like models presented above, where the number of independent fields depends on the number of blocks in the metric. As a concrete example we will consider a model which splits into two blocks of sizes $k \times k$ and $l \times l$, with $k \geq 2$, as the generalisation to the case of more blocks is straight-forward². The first thing to do is to set all scalar fields within each block proportional to one another

$$\sigma^0 \propto \dots \propto \sigma^{k-1}, \quad \sigma^k \propto \dots \propto \sigma^n,$$

which implies that the dual coordinates σ_I are also proportional to one another

$$\sigma_{(0)} := \sigma_0 \propto \dots \propto \sigma_{k-1}, \quad \sigma_{(1)} := \sigma_k \propto \dots \propto \sigma_n.$$

This identification is motivated by the fact that if one takes the solution to *STU*-like models (5.22) as an ansatz then the scalar fields in each block must be proportional to one another [60]. We have labelled the two independent fields by $\sigma_{(0)}$ and $\sigma_{(1)}$, in terms of which all other scalar fields can be deduced. We correspondingly fix the ratio between the charges in each block by setting

$$Q_{(0)} := Q_0 = \frac{\sigma_1}{\sigma_0} Q_1 = \dots = \frac{\sigma_{k-1}}{\sigma_0} Q_{k-1},$$

$$Q_{(1)} := Q_k = \frac{\sigma_{k+1}}{\sigma_k} Q_{k+1} = \dots = \frac{\sigma_n}{\sigma_k} Q_n.$$

Since the first block contains h^0 and has size $k \geq 2$ the physical scalar fields in the first block are all constant

$$\phi^1 = \frac{\sigma^1}{\sigma^2} \phi^2 = \dots = \frac{\sigma^1}{\sigma^{k-1}} \phi^{k-1} = \text{const}.$$

There is only one independent physical scalar field corresponding to the second block

$$\phi^{(1)} := \phi^k = \frac{\sigma^k}{\sigma^{k+1}} \phi^{k+1} = \dots = \frac{\sigma^k}{\sigma^n} \phi^n.$$

After making these identifications between the fields in each block, the equations of motion simplify to

$$\ddot{\sigma}_{(0)} - \frac{[(\dot{\sigma}_{(0)})^2 - (Q_{(0)})^2]}{\sigma_{(0)}} = 0, \quad (5.27)$$

$$\ddot{\sigma}_{(1)} - \frac{[(\dot{\sigma}_{(1)})^2 - (Q_{(1)})^2]}{\sigma_{(1)}} = 0, \quad (5.28)$$

$$\psi_0 \frac{[(\dot{\sigma}_{(0)})^2 - (Q_{(0)})^2]}{\sigma_{(0)}^2} + \psi_1 \frac{[(\dot{\sigma}_{(1)})^2 - (Q_{(1)})^2]}{\sigma_{(1)}^2} = 4c^2, \quad (5.29)$$

where ψ_0 and ψ_1 are constants depending on the particular ratios between the scalar fields in each block, and must satisfy $\psi_0 + \psi_1 = 1$. The second order equations (5.27) and (5.28) can be integrated to find

$$\sigma_{(0)} = \pm \frac{Q_{(0)}}{B_{(0)}} \sinh \left(B_{(0)} \tau + B_{(0)} \frac{A_{(0)}}{Q_{(0)}} \right), \quad (5.30)$$

$$\sigma_{(1)} = \pm \frac{Q_{(1)}}{B_{(1)}} \sinh \left(B_{(1)} \tau + B_{(1)} \frac{A_{(1)}}{Q_{(1)}} \right). \quad (5.31)$$

²If a model does not have at least one block of size 2×2 then it automatically falls into the class of *STU*-like models already discussed.

Substituting these solutions into (5.29) places one constraint on the $B_{(0,1)}$ integration constants

$$\psi_0 B_{(0)}^2 + \psi_1 B_{(1)}^2 = 4c^2 . \quad (5.32)$$

Let us now lift this solution to five dimensions. First, the KK-scalar is given through (4.9) by

$$e^{-\tilde{\sigma}} = \mu (\sigma_{(0)})^{\frac{k}{(n+1)}} (\sigma_{(1)})^{\frac{l}{(n+1)}} ,$$

where μ is a constant that depends on the ratios between the scalar fields. The requirement that the solution is asymptotically Minkowski places one constraint on $A_{(0,1)}$. The line element is given through (4.7) by

$$ds_5^2 = - \frac{1}{\mu (\sigma_{(0)})^{\frac{2k}{(n+1)}} (\sigma_{(1)})^{\frac{2l}{(n+1)}}} dt^2 \\ + \mu (\sigma_{(0)})^{\frac{k}{(n+1)}} (\sigma_{(1)})^{\frac{l}{(n+1)}} \left(\frac{c^3}{\sinh^3 2c\tau} d\tau^2 + \frac{c}{\sinh 2c\tau} d\Omega_{(3)}^2 \right) .$$

We must now check if this describes a black hole. The area of the horizon is given by

$$A = \lim_{\tau \rightarrow +\infty} \mu (\sigma_{(0)})^{\frac{k}{(n+1)}} (\sigma_{(1)})^{\frac{l}{(n+1)}} \frac{c}{\sinh 2c\tau} .$$

In order for the limit to converge the highest order term in the numerator, given by $\exp\left(\left(\frac{k}{(n+1)}B_{(0)} + \frac{l}{(n+1)}B_{(1)}\right)\tau\right)$, must exactly cancel with the highest order term in the denominator, given by $\exp(2c\tau)$. This condition is only satisfied if

$$\frac{k}{(n+1)}B_{(0)} + \frac{l}{(n+1)}B_{(1)} = 2c .$$

The single independent physical scalar field $\phi^{(1)}$ is given by

$$\phi^{(1)} \sim \frac{\sigma_{(0)}}{\sigma_{(1)}} .$$

The physical scalars therefore only take finite values on the horizon if $B_{(0)} = B_{(1)}$. Combined with the requirement that the area is finite we conclude that

$$B_{(0)} = B_{(1)} = 2c .$$

Collecting everything together, we find that the full solution for our example of a metric with two blocks is given by

$$ds_5^2 = - \frac{W}{(\mathcal{H}_{(0)})^{\frac{2k}{(n+1)}} (\mathcal{H}_{(1)})^{\frac{2l}{(n+1)}}} dt^2 + (\mathcal{H}_{(0)})^{\frac{k}{(n+1)}} (\mathcal{H}_{(1)})^{\frac{l}{(n+1)}} \left(\frac{dr^2}{W} + r^2 d\Omega_{(3)}^2 \right) ,$$

where

$$W = 1 - \frac{2c}{r^2} , \quad \mathcal{H}_{(0,1)} = \pm \mu \left[\frac{Q_{(0,1)}}{2c} \sinh \left(2c \frac{A_{(0,1)}}{Q_{(0,1)}} \right) + \frac{Q_{(0,1)} e^{-2c \frac{A_{(0,1)}}{Q_{(0,1)}}}}{2} \frac{1}{r^2} \right] .$$

The scalar fields are given by

$$\phi^{(1)} \sim \frac{\sigma_{(0)}}{\sigma_{(1)}} , \quad \sigma_{(0,1)} = \frac{1}{\mu} \frac{\mathcal{H}_{(0,1)}}{\sqrt{W}} = \pm \frac{Q_{(0,1)}}{2c} \sinh \left(2c\tau + 2c \frac{A_{(0,1)}}{Q_{(0,1)}} \right) .$$

Setting the size of the second block to zero corresponds to taking $l = 0$, $k = (n+1)$ and $\mu = \psi_0 = 1$. In this case the solution reduces simply to the five-dimensional non-extremal Reissner–Nordström solution [60]. Since all metrics contain at least one block this provides a universal solution with constant scalar fields for all models.

Chapter 6

The c-map

In this chapter we reduce a theory of $4d, \mathcal{N} = 2$ supergravity coupled to an arbitrary number of vector multiplets over a timelike or spacelike dimension. This induces a map from the target manifold of the four-dimensional theory to the target manifold of the three-dimensional theory called the c-map. In the case of reduction over space the target manifold in the image of the c-map is a quaternion Kähler manifold [26]. We will present the proof, for the first time, that in the case of reduction over time the target manifold in the image of the c-map is para-quaternion Kähler. This will appear in a later publication [107].

We will actually present a new formulation of the c-map that is manifestly symplectically covariant and presented entirely in terms of real Darboux coordinates. In order to do so we must first reformulate the geometry of the target manifold before reduction, i.e. projective special Kähler geometry, in terms of real coordinates. This is a useful result in its own right, as this formulation of projective special Kähler geometry is symplectically covariant, so long as a $U(1)$ gauge symmetry is not fixed. It also gives us another tool with which to study projective special Kähler geometry. This section is based on the publication [19] by the author.

We will perform the dimensional reduction of $4d, \mathcal{N} = 2$ supergravity coupled to an arbitrary number of vector multiplets over a timelike or spacelike dimension. This is formulated initially using the standard special complex coordinates. We then make a coordinate redefinition, analogous to the r-map, where we define new coordinates that absorb a degree of freedom from the metric to lift a hypersurface constraint (the D-gauge). Only then do we formulate the resulting three-dimensional Lagrangian in terms of special real coordinates. We then investigate the geometry of the target manifold of the reduction over a spacelike and timelike dimension separately. We first recover the known result for spacelike reduction using the real formulation, before proving the new result in the timelike case. Using our formalism we also prove that two integrable and metric compatible complex structures exist on the target manifold of both the spacelike and timelike c-map. One of these structures is associated with the quaternion structure after spacelike reduction, while the other is associated with the para-quaternion structure of timelike reduction.

We review the real formulation of affine special Kähler geometry in 6.1.1, before providing a new real formulation of projective special Kähler geometry in 6.1.2. We then perform the dimensional reduction of $4d, \mathcal{N} = 2$ supergravity coupled to vector multiplets in section 6.2 using complex coordinates, and then formulate the resulting target manifold geometry in terms of real coordinates. In section 6.3 we perform some necessary technical calculations: we construct a basis of real vielbein one-forms on

the co-tangent space of the target manifold both before and after reduction, and then calculate the exterior derivatives of these one-forms. We use these calculations to recover the results of Ferrara and Sabharwal in section 6.4, and go further to prove that the timelike c-map has a para-quaternion Kähler target manifold in section 6.5.

6.1 Real formulation of special Kähler geometry

It is useful both physically and mathematically to consider the real formulation of projective special Kähler geometry.

Physically it means that fields transform in a simpler way under symplectic transformations, as special real coordinates transform as a vector and the metric in real coordinates transforms as a tensor. It is therefore much easier to formulate special Kähler geometry and the c-map in a symplectically covariant way. This, in turn, greatly simplifies the task of finding stationary solutions to $4D$, $\mathcal{N} = 2$ supergravity.

Mathematically, the real formulation of affine special Kähler and projective special Kähler geometry provides an alternative framework for studying the geometry of these manifolds, which is often clearer than the standard complex formulation. It also has clear advantages when considering the c-map. For the standard spacelike c-map it will help expose some interesting new features, such as the existence of an additional integrable complex structure. For the timelike c-map we will use it to prove that the target manifold is be para-quaternion Kähler, and there exist two integrable complex structures.

6.1.1 Real formulation of affine special Kähler geometry

We begin by decomposing a special holomorphic coordinate system X^I and conjugate coordinate system F_I into real and imaginary parts

$$\begin{aligned} X^I &= x^I + iu^I(x, y) , \\ F_I &= y_I + iv_I(x, y) . \end{aligned}$$

Recall from remark 2.3 that the real functions x^I, y_I form a flat Darboux coordinate system, which we shall call a special real coordinate system. We will now formulate the properties of affine special Kähler geometry in terms of these real coordinates. It will be useful to label x^I, y_I as a single set of coordinates

$$q^a := (x^I, y_I)^T ,$$

which has the advantage that q^a transforms linearly under symplectomorphisms. Affine special Kähler manifolds admit not just one special connection but a whole S^1 family [11]. Parametrising S^1 by α , the family is generated by the transformation

$$\nabla^{(\alpha)} = e^{\alpha I} \circ \nabla \circ e^{-\alpha I} .$$

Each connection in this family has its own adapted set of special real coordinates, given by the real parts of $e^{i\alpha}(X^I, F_I)$. Since neither physics nor geometry depend on the choice of special connection we may just as well take any special real coordinates in this family. For example we could take $(u^I, v_I)^T$ as special real coordinates, which are flat Darboux coordinates with respect to the $\nabla^{(-\pi/2)}$ special connection.

We would like to determine a formula for the metric in the special real coordinate system, i.e. find a matrix H_{ab} such that

$$g_N = \text{Re} (N_{IJ} dX^I \otimes d\bar{X}^J) = H_{ab} dq^a \otimes dq^b . \quad (6.1)$$

Proposition 6.1 (Freed). The metric H_{ab} is the Hessian of a real function H .

Proof. Since special real coordinates q^a are Darboux coordinates the coefficients of the symplectic form Ω are constant

$$\Omega = \frac{1}{2} \Omega_{ab} dq^a \wedge dq^b , \quad \Omega_{ab} := \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} .$$

The symplectic form is related to the metric by $\Omega = \frac{1}{2} g(J \cdot, \cdot)$, which we can express in components as

$$H_{ab} = 2\Omega_{ac} J^c_b . \quad (6.2)$$

We now take derivatives of the metric coefficients with respect to q^d , and by making use of (6.2) and the fact that $\nabla_X(JY) = \nabla_Y(JX)$ we get

$$\frac{\partial}{\partial q^d} H_{ab} = 2\Omega_{ac} \frac{\partial}{\partial q^d} J^c_b = 2\Omega_{ac} \frac{\partial}{\partial q^b} J^c_d = \frac{\partial}{\partial q^b} H_{ad} .$$

It follows from the standard theory of Hessian structures that we can locally write the metric coefficients as

$$H_{ab} = \frac{\partial^2 H}{\partial q^a \partial q^b} ,$$

for some real function H [70]. □

The Hesse potential is related to the holomorphic prepotential by a Legendre transformation that replaces u^I by y_I as independent variables [108]

$$H(x, y) = 2\text{Im}F(X(x, y)) - 2y_I u^I(x, y) , \quad (6.3)$$

and is also proportional to the Kähler potential [11]

$$-2H(x, y) = -i (X^I(x, y) \bar{F}_I(x, y) - F_I(x, y) \bar{X}^I(x, y)) . \quad (6.4)$$

We need to take second derivatives of the Hesse potential in order to find the metric coefficients H_{ab} . Taking derivatives of the first term of (6.3) we find

$$\begin{aligned} \frac{\partial}{\partial x^I} 2\text{Im}(F) \Big|_{x, u(x, y)} &= \left(\frac{\partial}{\partial x^I} + \frac{\partial u^J}{\partial x^I} \frac{\partial}{\partial u^J} \right) 2\text{Im}(F) \Big|_{x, u} \\ &= \left[\left(\frac{\partial}{\partial X^I} + \frac{\partial}{\partial \bar{X}^I} \right) + i \frac{\partial u^J}{\partial x^I} \left(\frac{\partial}{\partial X^J} - \frac{\partial}{\partial \bar{X}^J} \right) \right] 2\text{Im}(F) \Big|_{X, \bar{X}} \\ &= 2v_I + 2y_I \frac{\partial u^J}{\partial x^I} , \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial y_I} 2\text{Im}(F) \Big|_{x, u(x, y)} &= \left(\frac{\partial u^J}{\partial y_I} \frac{\partial}{\partial u^J} \right) 2\text{Im}(F) \Big|_{x, u} \\ &= i \frac{\partial u^J}{\partial y_I} \left(\frac{\partial}{\partial X^J} - \frac{\partial}{\partial \bar{X}^J} \right) 2\text{Im}(F) \Big|_{X, \bar{X}} \\ &= 2y_J \frac{\partial u^J}{\partial y_I} . \end{aligned}$$

Using these results, we find that the first derivatives of the Hesse potential are proportional to u^I, v_I :

$$H_a := \frac{\partial H}{\partial q^a} = \left(\frac{\partial H}{\partial x^I}, \frac{\partial H}{\partial y_I} \right) = \begin{pmatrix} 2v_I \\ -2u^I \end{pmatrix}. \quad (6.5)$$

Before we can take second derivatives of H we need to prove some identities. First, from the fact that $\partial\bar{\partial}F = 0$ we have

$$(dx^I + idu^I) \wedge (dy_I + idv_I) = 0.$$

The imaginary part of the above expression tells us that

$$\begin{aligned} \frac{\partial v_I}{\partial x^J} &= \frac{\partial v_J}{\partial x^I}, \\ \frac{\partial v_I}{\partial y_J} &= -\frac{\partial u^J}{\partial x^I}, \\ \frac{\partial u^I}{\partial y_J} &= \frac{\partial u^J}{\partial y_I}. \end{aligned}$$

Next, by computing the Jacobian of the transformation $(X, \bar{X}) \leftrightarrow (x, u)$ we have

$$\begin{aligned} \frac{\partial v_I}{\partial u^J} &= \left(\frac{\partial}{\partial X^J} + \frac{\partial}{\partial \bar{X}^J} \right) \frac{1}{2i} (F_I - \bar{F}_I) = \frac{1}{2} R_{IJ}, \\ \frac{\partial y_I}{\partial u^J} &= \left(\frac{\partial}{\partial X^J} + \frac{\partial}{\partial \bar{X}^J} \right) \frac{1}{2} (F_I + \bar{F}_I) = -\frac{1}{2} N_{IJ}, \\ \frac{\partial u^I}{\partial x^J} &= -\frac{\partial v_I}{\partial y_J} = -\frac{\partial u^K}{\partial y_J} \frac{\partial v_I}{\partial u^K} = N^{IK} R_{KJ}. \end{aligned}$$

Using these expression we can compute the second derivatives of H

$$\begin{aligned} \frac{\partial^2 H}{\partial x^I \partial x^J} &= N_{IJ} + R_{IK} N^{KL} R_{LJ}, \\ \frac{\partial^2 H}{\partial x^I \partial y_J} &= -2N^{JK} R_{KI}, \\ \frac{\partial^2 H}{\partial y_I \partial y_J} &= 4N^{IJ}. \end{aligned}$$

We can therefore write the components of the metric in terms of N_{IJ} and R_{IJ} as

$$H_{ab} = \begin{pmatrix} N + RN^{-1}R & -2RN^{-1} \\ -2N^{-1}R & 4N^{-1} \end{pmatrix}. \quad (6.6)$$

It is also useful to note the relation between the differentials of the special holomorphic and the special real coordinates:

$$\begin{aligned} dX^M &= dx^M + i \left(\frac{\partial u^M}{\partial x^K} dx^K + \frac{\partial u^M}{\partial y_I} dy_I \right) \\ &= dx^M + i \left(N^{MI} R_{IK} dx^K - 2N^{MI} dy_I \right). \end{aligned} \quad (6.7)$$

Using equations (6.6) and (6.7) it is straightforward to verify that H_{ab} satisfies (6.1), which shows that N_{IJ} and H_{ab} represent the same metric in terms of special holomorphic and special real coordinates respectively. The inverse metric is given by

$$H^{ab} = \begin{pmatrix} N^{-1} & \frac{1}{2}N^{-1}R \\ \frac{1}{2}RN^{-1} & \frac{1}{4}(N + RN^{-1}R) \end{pmatrix}.$$

Using special real coordinates has the advantage that the metric is manifestly symplectically covariant: H_{ab} transforms as a rank two tensor while dq^a transforms as a covector. This is in contrast with F_{IJ} , which transforms fractionally linearly [11].

We end this section by noting that since $\Omega = \frac{1}{2}gJ$ it follows that

$$\Omega^{-1} = 2J^{-1}g^{-1} = -2Jg^{-1} \quad \Rightarrow \quad g\Omega^{-1}g = -2gJg^{-1}g = -4\Omega ,$$

which we can write in components as

$$H_{ab}\Omega^{bc}H_{cd} = -4\Omega_{ad} . \quad (6.8)$$

This also gives us a formula for the complex structure

$$J = -\frac{1}{2}\Omega^{-1}H = -\frac{1}{2}\Omega^{ab}H_{bc}\partial_a \otimes dq^c . \quad (6.9)$$

6.1.2 Real formulation of projective special Kähler geometry

We are now ready to formulate projective special Kähler geometry in terms of special real coordinates. Rather than work at the level of the projective special Kähler manifold, we prefer to work with horizontal fields on the corresponding conic affine special Kähler manifold. This is in contrast to the real formulation presented in [22], which works exclusively at the level of the projective special Kähler manifold. We will see that the benefit of our approach is that full symplectic covariance is kept manifest, so long as the $U(1)$ symmetry is not fixed. Our aim will be to write the real part of the tensor field

$$g_{I\bar{J}}dX^I \otimes d\bar{X}^J = \left(-\frac{N_{IJ}}{\bar{X}NX} + \frac{(N\bar{X})_I(NX)_J}{(\bar{X}NX)^2} \right) dX^I \otimes d\bar{X}^J$$

in terms of special real coordinates $q^a = (x^I, y_I)^T$. This tensor field can be projected down to give the metric on the projective special Kähler manifold. Throughout this section let \bar{N} denote a positive definite projective special Kähler manifold, and N the corresponding conic affine special Kähler manifold.

We begin by noting that for a conic affine special Kähler manifold the Hesse potential is a homogeneous function of degree two, and so we have

$$H_{ab}q^b = H_a , \quad H_{ab}q^aq^b = 2H , \quad q^a = H^{ab}H_b . \quad (6.10)$$

Using the above expressions along with (6.8) one may calculate the following useful identities:

$$H_{ab}\Omega^{bc}H_c = -4\Omega_{ab}q^b , \quad (6.11a)$$

$$H^{ab}H_{bcd}\Omega^{de} = -\Omega^{ab}H_{bcd}H^{de} . \quad (6.11b)$$

Recall that on a conic affine special Kähler manifold we have a homothetic Killing vector field ξ and Killing vector field $J\xi$. These are written in special real coordinates as

$$\begin{aligned} \xi &= q^a \frac{\partial}{\partial q^a} , \\ J\xi &= \frac{1}{2}H_a\Omega^{ab}\frac{\partial}{\partial q^b} . \end{aligned}$$

The next step is to find an expression for the matrix $g_{I\bar{J}}$ in terms of special real coordinates. Recall from (6.4) that the Hesse potential is related to the Kähler potential by $-2H = e^{-\mathcal{K}}$, where \mathcal{K} is defined as in section 2.3. Since $g_{I\bar{J}} = \partial_{I,\bar{J}}^2 \mathcal{K}$ we can write this matrix as

$$g_{I\bar{J}} = -\frac{1}{H} \frac{\partial^2 H}{\partial X^I \partial \bar{X}^J} + \frac{1}{H^2} \frac{\partial H}{\partial X^I} \frac{\partial H}{\partial \bar{X}^J}, \quad (6.12)$$

which leaves us to calculate the derivatives of the Hesse potential with respect to \bar{X}, X . The first derivatives are calculated to be

$$\begin{aligned} \frac{\partial H}{\partial X^I} &= \frac{\partial x^J}{\partial X^I} \frac{\partial H}{\partial x^J} + \frac{\partial y_J}{\partial X^I} \frac{\partial H}{\partial y_J} \\ &= 2v_J \frac{\partial x^J}{\partial X^I} - 2u^J \frac{\partial y_J}{\partial X^I} \\ &= v_I - \frac{1}{2} (R_{IJ} + iN_{IJ}) u^J, \end{aligned} \quad (6.13)$$

and by a similar calculation

$$\frac{\partial H}{\partial \bar{X}^I} = v_I - \frac{1}{2} (R_{IJ} - iN_{IJ}) u^J. \quad (6.14)$$

We can then calculate the second derivatives of the Hesse potential

$$\frac{\partial^2 H}{\partial \bar{X}^J \partial X^I} = \frac{1}{2} N_{IJ}. \quad (6.15)$$

Substituting (6.13), (6.14) and (6.15) into (6.12) we are left with the expression

$$g_{I\bar{J}} = -\frac{1}{2H} N_{IJ} + \frac{1}{H^2} \left(v_I - \frac{1}{2} (R_{IK} + iN_{IK}) u^K \right) \left(v_J - \frac{1}{2} (R_{JL} - iN_{JL}) u^L \right). \quad (6.16)$$

We shall now consider the real part of (6.16) contracted with $dX^I \otimes d\bar{X}^J$. The first term can be deduced simply from (6.1). For the second term we first use (6.10) and (6.5) to write

$$\begin{aligned} x^I &= 2N^{IJ} v_J - N^{IJ} R_{JK} u^K, \\ y_I &= R_{IJ} N^{JK} v_K - \frac{1}{2} (N_{IJ} + R_{IK} N^{KL} R_{LJ}) u^J. \end{aligned}$$

Using this together with (6.7) we find that

$$\begin{aligned} \text{Re} \Big\{ & \left(v_I - \frac{1}{2} (R_{IK} + iN_{IK}) u^K \right) \left(v_J - \frac{1}{2} (R_{JL} - iN_{JL}) u^L \right) dX^I \otimes d\bar{X}^J \Big\} \\ &= (v_I v_J + y_I y_J) dx^I \otimes dx^J - (v_I u^J + y_I x^J) dx^I \otimes dy_J \\ &\quad - (u^I v_J + x^I y_J) dy_I \otimes dx^J + (u^I u^J + x^I x^J) dy_I \otimes dy_J. \end{aligned}$$

This expression can be simplified by making the observation that

$$H_a H_b = 4 \begin{pmatrix} v_I v_J & -v_I u^J \\ -u^I v_J & u^I u^J \end{pmatrix},$$

and

$$\Omega_{ac} q^c \Omega_{bd} q^d = \begin{pmatrix} y_I y_J & -y_I x^J \\ -x^I y_J & x^I x^J \end{pmatrix}.$$

We then have the simple expression

$$\begin{aligned} & \frac{1}{H^2} \left((v_I v_J + y_I y_J) dx^I \otimes dx^J - (v_I u^J + y_I x^J) dx^I \otimes dy_J \right. \\ & \quad \left. - (u^I v_J + x^I y_J) dy_I \otimes dx^J + (u^I u^J + x^I x^J) dy_I \otimes dy_J \right) \\ & = \left(\frac{1}{4H^2} H_a H_b + \frac{1}{H^2} \Omega_{ac} q^c \Omega_{bd} q^d \right) dq^a \otimes dq^b . \end{aligned}$$

Putting everything together, by defining the matrix

$$H_{ab}^{(0)} := -\frac{1}{2H} H_{ab} + \frac{1}{4H^2} H_a H_b + \frac{1}{H^2} \Omega_{ac} q^c \Omega_{bd} q^d ,$$

we finally obtain the desired expression:

$$\operatorname{Re} (g_{I\bar{J}} dX^I \otimes d\bar{X}^J) = H_{ab}^{(0)} dq^a \otimes dq^b . \quad (6.17)$$

One can interpret $H_{ab}^{(0)}$ as the horizontal lift of the projective special Kähler metric, expressed in terms of special real coordinates. It is straightforward to check that $H_{ab}^{(0)}$ is degenerate along the directions ξ and $J\xi$.

Let us take a moment to analyse expression (6.17) for the projective special Kähler metric in terms of real coordinates. The one-forms dual to the vector fields ξ and $J\xi$ are given by

$$\xi^\flat = \frac{1}{2H} H_a dq^a , \quad (J\xi)^\flat = \frac{1}{H} dq^a \Omega_{ac} q^c ,$$

and plugging ξ into the metric on the conic affine special Kähler manifold gives $g_N(\xi, \xi) = g_N(J\xi, J\xi) = 2H$. We can therefore write the expression (6.17) in a coordinate free way as

$$H^{(0)} = -\frac{g_N}{g_N(\xi, \xi)} + \xi^\flat \otimes \xi^\flat + (J\xi)^\flat \otimes (J\xi)^\flat .$$

The meaning of each term now becomes clear. The first term is a rescaling of the metric, which turns the homothety ξ into an isometry, as can be seen by

$$\mathcal{L}_\xi \left(\frac{g_N}{g_N(\xi, \xi)} \right) = \frac{\mathcal{L}_\xi g_N}{g_N(\xi, \xi)} - \frac{g_N}{g_N(\xi, \xi)^2} (\mathcal{L}_\xi g_N)(\xi, \xi) = 0 ,$$

where we used the fact that $\mathcal{L}_\xi g_N = 2g_N$. This ensures that it can be projected down to give the metric on the projective special Kähler manifold. The second term ensures that the $H^{(0)}$ is degenerate along the ξ direction, and the third term ensures it is also degenerate along the $J\xi$ direction. It is therefore clear that $H^{(0)}$ corresponds to the horizontal lift of the metric on the projective special Kähler manifold \bar{N} .

At this point it is useful to introduce a new tensor field by first defining the function

$$\tilde{H} := -\frac{1}{2} \log(2H) , \quad (6.18)$$

and then taking second derivatives

$$\tilde{H}_{ab} := \partial_{a,b}^2 \tilde{H} .$$

This tensor is a non-degenerate rank two tensor field on N , and can therefore be interpreted as a new Hessian metric. It is negative definite along all directions except

ξ , where it is positive definite. Using the homogeneity properties of H we establish the useful identity

$$\tilde{H}_{ab} q^a q^b = 1 . \quad (6.19)$$

We can write the tensor $H^{(0)}$ in terms of \tilde{H}_{ab} as

$$H_{ab}^{(0)} = \tilde{H}_{ab} - \frac{1}{4H^2} H_a H_b + \frac{1}{H^2} (\Omega_{ac} q^c) (\Omega_{bd} q^d) ,$$

and so we have

$$\text{Re} (g_{IJ} dX^I \otimes d\bar{X}^J) = \left[\tilde{H}_{ab} - \frac{1}{4H^2} H_a H_b + \frac{1}{H^2} (\Omega_{ac} q^c) (\Omega_{bd} q^d) \right] dq^a \otimes dq^b . \quad (6.20)$$

We now turn our attention to the coupling matrix of the vector kinetic terms, which can be written as (3.7)

$$\mathcal{N}_{IJ} = \mathcal{R}_{IJ} + i\mathcal{I}_{IJ} = \bar{F}_{IJ} + i \frac{(NX)_I (NX)_J}{XNX} . \quad (6.21)$$

These matrices give the couplings of the gauge field terms in the Lagrangian of four-dimensional supergravity coupled to vector multiplets, and will play an important role later on when we consider the c-map. Rather than trying to calculate the matrix \mathcal{N}_{IJ} in terms of special real coordinates, which does not give a particularly nice expression, we will instead focus on the matrix

$$\hat{H}_{ab} := \begin{pmatrix} \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1} \\ -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \end{pmatrix} . \quad (6.22)$$

This matrix appears as the coupling matrix of the vector kinetic terms after dimensional reduction, and we will see that it can be written simply in terms of special real coordinates. It is a non-degenerate rank two matrix on N and can therefore be interpreted as a metric (which is negative definite). Note that so far we have introduced three different metrics on N , which have signatures

$$\begin{array}{ccc} H_{ab} & \tilde{H}_{ab} & \hat{H}_{ab} \\ \underbrace{(-}_{\xi} \underbrace{-}_{J\xi} \underbrace{+\dots+}_{\{\xi, J\xi\}^\perp} & (+ - - \dots -) & (- - - \dots -) . \end{array}$$

Proposition 6.2 . The three metrics H_{ab} , \tilde{H}_{ab} and \hat{H}_{ab} are related through the expression

$$\begin{aligned} \tilde{H}_{ab} &= -\frac{1}{2H} H_{ab} + \frac{1}{2H^2} H_a H_b \\ &= \frac{1}{H} \hat{H}_{ab} - \frac{2}{H^2} (\Omega_{ac} q^c) (\Omega_{bd} q^d) . \end{aligned} \quad (6.23)$$

Proof. We begin by writing down the explicit relations between the real and imaginary parts of $F_{IJ} = \frac{1}{2}(R_{IJ} + iN_{IJ})$ and the real and imaginary parts of $\mathcal{N}_{IJ} = \mathcal{R}_{IJ} + i\mathcal{I}_{IJ}$:

$$\begin{aligned} \mathcal{R}_{IJ} &= \frac{1}{2} R_{IJ} + \frac{1}{2} \left(\frac{(NX)_I (NX)_J}{XNX} - \frac{(N\bar{X})_I (N\bar{X})_J}{\bar{X}N\bar{X}} \right) , \\ \mathcal{I}_{IJ} &= -\frac{1}{2} N_{IJ} + \frac{1}{2} \left(\frac{(NX)_I (NX)_J}{XNX} - \frac{(N\bar{X})_I (N\bar{X})_J}{\bar{X}N\bar{X}} \right) . \end{aligned}$$

We will now write every term in the block decomposition of \hat{H}_{IJ} given by expression (6.22) in terms of special real coordinates. First, the inverse of \mathcal{I}_{IJ} is given by

$$\begin{aligned}\mathcal{I}^{IJ} &= -2N^{IJ} + \frac{2}{(XN\bar{X})} (X^I \bar{X}^J + \bar{X}^I X^J) \\ &= -2N^{IJ} + \frac{2}{H} (x^I x^J + u^I u^J) ,\end{aligned}$$

where we used $2H = \bar{X}NX$. This provides the bottom right term. Contracting this with \mathcal{R}_{IJ} we find

$$\begin{aligned}-\mathcal{I}^{IK} \mathcal{R}_{KJ} &= N^{IK} R_{KJ} - \frac{2}{(XN\bar{X})} (X^I \bar{F}_J + \bar{X}^I F_J) \\ &= N^{IK} R_{KJ} - \frac{2}{H} (x^I y_J + u^I v_J) ,\end{aligned}$$

which provides the upper-right and bottom-left terms. Finally, for the upper-left term we have

$$\begin{aligned}\mathcal{I}_{IJ} + \mathcal{R}_{IK} \mathcal{I}^{KL} \mathcal{R}_{LJ} &= -\frac{1}{2} N_{IJ} - \frac{1}{2} R_{IK} N^{KL} R_{LJ} + \frac{2}{(XN\bar{X})} (F_I \bar{F}_J + \bar{F}_I F_J) \\ &= -\frac{1}{2} N_{IJ} - \frac{1}{2} R_{IK} N^{KL} R_{LJ} + \frac{2}{H} (y_I y_J + v_I v_J) .\end{aligned}$$

Putting everything together we find

$$\begin{aligned}\begin{pmatrix} \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1} \\ -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \end{pmatrix} &= \begin{pmatrix} -\frac{1}{2}N - \frac{1}{2}RN^{-1}R & RN^{-1} \\ RN^{-1} & -2N^{-1} \end{pmatrix} \\ &\quad + \frac{2}{H} \begin{pmatrix} y_I y_J + v_I v_J & -(y_I x^J + v_I u^J) \\ -(x^I y_J + u^I v_J) & x^I x^J + u^I u^J \end{pmatrix} ,\end{aligned}$$

which we can write as

$$\hat{H}_{ab} = -\frac{1}{2} H_{ab} + \frac{2}{H} \left(\frac{1}{4} H_a H_b + \Omega_{ac} q^c \Omega_{bd} q^d \right) .$$

From the definition of \tilde{H}_{ab} we also have

$$\tilde{H}_{ab} = -\frac{1}{2H} H_{ab} + \frac{1}{2H^2} H_a H_b ,$$

which completes the proof. □

6.2 Dimensional reduction

Our starting point is the bosonic part of the Lagrangian of four-dimensional $\mathcal{N} = 2$ supergravity coupled to n vector multiplets (with Lorentzian signature)

$$\mathbf{e}_4^{-1} \mathcal{L}_4 = \frac{1}{2} R_4 - g_{A\bar{B}} \partial_{\hat{\mu}} z^A \partial^{\hat{\mu}} \bar{z}^{\bar{B}} + \frac{1}{4} \mathcal{I}_{IJ} \hat{F}_{\hat{\mu}\hat{\nu}}^I \hat{F}^{J\hat{\mu}\hat{\nu}} + \frac{1}{4} \mathcal{R}_{IJ} \hat{F}_{\hat{\mu}\hat{\nu}}^I \tilde{\hat{F}}^{J\hat{\mu}\hat{\nu}} . \quad (6.24)$$

Note that $g_{A\bar{B}}$ is positive definite and \mathcal{I}_{IJ} is negative definite, which ensures that both the scalars and gauge fields have the standard sign in front of their kinetic terms. As in the case of the r-map, we prefer not to work with the coordinates z^A , which parametrise

the physical co-dimension two hypersurface, but rather the special coordinates X^I , which live in the larger ambient space. To describe the same physics we must at some point restrict X^I to the physical hypersurface by imposing two real constraints, which fix dilatations and $U(1)$ phase transformations. Since these are symmetries of the Lagrangian this may be done before or after calculating the equations of motion. We will fix the dilatations immediately by imposing the so-called D-gauge

$$-i(X^I \bar{F}_I - F_I \bar{X}^I) = 1. \quad (6.25)$$

For the $U(1)$ symmetry we take a different approach. We could fix this symmetry by applying any appropriate condition that fixes the overall phase of the X^I coordinates, e.g.

$$\text{Im}(X^0) = 0, \quad (6.26)$$

however we will choose not to fix this symmetry until we solve the equations of motion. The reason, as we shall see shortly, is that any $U(1)$ gauge fixing condition that involves just the X^I coordinates will break the symplectic covariance of the Lagrangian. However, one may impose a symplectically covariant gauge fixing condition that involves both the X^I coordinates and certain components of the gauge fields, but since this requires imposing part of the field equations we postpone this discussion until later. In the Lagrangian we must make the replacement

$$g_{A\bar{B}}(z) \partial_{\hat{\mu}} z^A \partial^{\hat{\mu}} \bar{z}^B \longrightarrow g_{I\bar{J}}(X, \bar{X}) \partial_{\hat{\mu}} X^I \partial^{\hat{\mu}} \bar{X}^J.$$

We can now write the Lagrangian (6.24) as

$$\mathbf{e}_4^{-1} \mathcal{L}_4 = \frac{1}{2} R_4 - g_{IJ} \partial_{\hat{\mu}} X^I \partial^{\hat{\mu}} \bar{X}^J + \frac{1}{4} \mathcal{I}_{IJ} \hat{F}_{\hat{\mu}\hat{\nu}}^I \hat{F}^{J\hat{\mu}\hat{\nu}} - \frac{1}{8} \mathcal{R}_{IJ} \hat{F}_{\hat{\mu}\hat{\nu}}^I \hat{F}_{\hat{\rho}\hat{\sigma}}^J \mathbf{e}_4^{-1} \hat{\epsilon}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}, \quad (6.27)$$

where it is understood that the X^I are subject to the constraint (6.25) and a $U(1)$ gauge fixing condition that will be imposed later. Here we have also written out the fourth term explicitly. The coupling matrices are given directly in terms of X^I, \bar{X}^I through the expressions (3.7) and (3.8).

We will reduce the Lagrangian (6.27) over a timelike or spacelike dimension simultaneously using the ϵ notation

$$\epsilon = \begin{cases} +1 & \text{timelike,} \\ -1 & \text{spacelike.} \end{cases}$$

As we shall see, the only difference between reduction over a timelike or spacelike dimension will appear in the signs in front of the various terms in the reduced Lagrangian. We begin by decomposing the metric in a manner adapted to the dimensional reduction procedure (see (3.20))

$$ds_4^2 = -\epsilon e^{\phi} (dy + V_{\mu} dx^{\mu})^2 + e^{-\phi} g_{\mu\nu} dx^{\mu} dx^{\nu}. \quad (6.28)$$

The three-dimensional metric $g_{\mu\nu}$ will have Lorentzian signature if we reduce over space or Euclidean signature if we reduce over time. For the gauge fields we make the decomposition (see (3.22))

$$\hat{A}^I = \zeta^I dy + (V_{\mu}^I - \zeta^I V_{\mu}) dx^{\mu},$$

where we have added a term proportional to V_{μ} to ensure that the reduced Lagrangian is manifestly invariant under three-dimensional gauge transformations [19]. We now

perform the dimensional reduction following the prescription outlined in section 3.3, and obtain the three-dimensional Lagrangian

$$\begin{aligned} \mathbf{e}_3^{-1} \mathcal{L}_3 = & \frac{1}{2} \left(R_3 - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{4} \epsilon e^{2\phi} V^{\mu\nu} V_{\mu\nu} \right) - g_{IJ} \partial_\mu X^I \partial^\mu \bar{X}^J \\ & + \frac{1}{4} e^\phi \mathcal{I}_{IJ} (F_{\mu\nu}^I + \zeta^I V_{\mu\nu}) (F^{J\mu\nu} + \zeta^J V^{\mu\nu}) \\ & - \frac{1}{2} \epsilon e^{-\phi} \mathcal{I}_{IJ} \partial_\mu \zeta^I \partial^\mu \zeta^J - \frac{1}{2} \mathcal{R}_{IJ} (F_{\mu\nu}^I + \zeta^I V_{\mu\nu}) \partial_\rho \zeta^J \mathbf{e}_3^{-1} \epsilon^{\mu\nu\rho} . \end{aligned} \quad (6.29)$$

In the first line we have collected together terms that descend from the Einstein–Hilbert term and scalar kinetic term. Note that the scalar kinetic term is completely unaffected by the reduction. Terms that descend from the gauge fields appear in the second and third lines.

The above Lagrangian encodes the dynamics of the theory in terms of a three-dimensional metric, scalar fields and gauge vectors. However, in three dimensions one can dualise gauge fields into scalar fields, and we can therefore write a dual Lagrangian that encodes the dynamics of the theory purely in terms of the metric and scalar fields. One can determine this dual Lagrangian by adding to (6.29) the Lagrange multiplier term

$$\mathbf{e}_3^{-1} \mathcal{L}_{\text{Lm}} = \frac{1}{2} \epsilon \varepsilon^{\mu\nu\rho} \left(F_{\mu\nu}^I \partial_\rho \tilde{\zeta}_I - V_{\mu\nu} \partial_\rho \left(\tilde{\phi} - \frac{1}{2} \zeta^I \tilde{\zeta}_I \right) \right) .$$

By taking the variation of $\tilde{\mathcal{L}}_3 := \mathcal{L}_3 + \mathcal{L}_{\text{Lm}}$ with respect to $V_{\mu\nu}$ and $F_{\mu\nu}^I$ one obtains the algebraic equations of motion

$$\begin{aligned} V_{\mu\nu} &= 2e^{-2\phi} \varepsilon_{\mu\nu\rho} \left(\partial^\rho \tilde{\phi} + \frac{1}{2} (\zeta^I \partial^\rho \tilde{\zeta}_I - \tilde{\zeta}_I \partial^\rho \zeta^I) \right) , \\ F_{\mu\nu}^I &= -\epsilon e^{-\phi} \mathcal{I}^{IJ} \varepsilon_{\mu\nu\rho} (\partial^\rho \tilde{\zeta}_J - \mathcal{R}_{JK} \partial^\rho \zeta^K) - \zeta^I V_{\mu\nu} . \end{aligned}$$

These algebraic equations can be substituted back into $\tilde{\mathcal{L}}_3$ to obtain the dual Lagrangian (dropping the tilde)

$$\begin{aligned} \mathbf{e}_3^{-1} \mathcal{L}_3 = & \frac{1}{2} R_3 - g_{IJ} \partial_\mu X^I \partial^\mu \bar{X}^J - \frac{1}{4} \partial_\mu \phi \partial^\mu \phi \\ & - e^{-2\phi} \left(\partial_\mu \tilde{\phi} + \frac{1}{2} (\zeta^I \partial_\mu \tilde{\zeta}_I - \tilde{\zeta}_I \partial_\mu \zeta^I) \right)^2 \\ & - \frac{1}{2} \epsilon e^{-\phi} \left[\mathcal{I}_{IJ} \partial_\mu \zeta^I \partial^\mu \zeta^J + \mathcal{I}^{IJ} \left(\partial_\mu \tilde{\zeta}_I - \mathcal{R}_{IK} \partial_\mu \zeta^K \right) \left(\partial^\mu \tilde{\zeta}_J - \mathcal{R}_{JL} \partial^\mu \zeta^L \right) \right] . \end{aligned} \quad (6.30)$$

The above three-dimensional Lagrangian is in the standard form that one often finds in the literature. The scalar fields describe a non linear sigma model with a quaternion Kähler target manifold equipped with the so-called Ferrara–Sabharwal metric [26]. However, inspired by our treatment of the r-map, we find it convenient to make a field redefinition. This will not only simplify the Lagrangian but also eliminates the need to impose the D-gauge. We must still impose a U(1) gauge or work with gauge invariant quantities. This is achieved by introducing the new complex scalar fields Y^I , defined by

$$Y^I := e^{\phi/2} X^I . \quad (6.31)$$

The constraint given by the D-gauge (6.25) now reads

$$-i(Y^I \bar{F}_I - F_I \bar{Y}^I) = e^\phi ,$$

which just gives an expression for the KK-scalar e^ϕ in terms of the new coordinates. Note that since the matrices \mathcal{I}_{IJ} and \mathcal{R}_{IJ} are homogeneous of degree zero it follows that

$$\mathcal{I}_{IJ}(X, \bar{X}) = \mathcal{I}_{IJ}(Y, \bar{Y}) , \quad \mathcal{R}_{IJ}(X, \bar{X}) = \mathcal{R}_{IJ}(Y, \bar{Y}) .$$

The matrix $g_{I\bar{J}}$ is homogeneous of degree -2 , and so

$$g_{I\bar{J}}(X, \bar{X}) = e^\phi g_{I\bar{J}}(Y, \bar{Y}) .$$

It follows that the Lagrangian takes the same form as before

$$\begin{aligned} \mathbf{e}_3^{-1} \mathcal{L}_3 = & \frac{1}{2} R_3 - g_{I\bar{J}}(Y, \bar{Y}) \partial_\mu Y^I \partial^\mu \bar{Y}^{\bar{J}} - \frac{1}{4} \partial_\mu \phi \partial^\mu \phi \\ & - e^{-2\phi} \left(\partial_\mu \tilde{\phi} + \frac{1}{2} (\zeta^I \partial_\mu \tilde{\zeta}_I - \tilde{\zeta}_I \partial_\mu \zeta^I) \right)^2 \\ & - \frac{1}{2} \epsilon e^{-\phi} \left[\mathcal{I}_{IJ} \partial_\mu \zeta^I \partial^\mu \zeta^J + \mathcal{I}^{IJ} \left(\partial_\mu \tilde{\zeta}_I - \mathcal{R}_{IK} \partial_\mu \zeta^K \right) \left(\partial^\mu \tilde{\zeta}_J - \mathcal{R}_{JL} \partial^\mu \zeta^L \right) \right] , \end{aligned} \quad (6.32)$$

however the fields Y^I are not constrained by the D-gauge, and the KK-scalar is a dependent field $\phi = \phi(Y, \bar{Y})$. There is still a U(1) symmetry of the Lagrangian that corresponds to overall phase transformations of the Y^I fields. As previously explained, this symmetry will be gauge fixed only after we have calculated the equations of motion.

6.2.1 The real formulation

We will now rewrite the Lagrangian (6.32) in terms of the special real coordinates we introduced in the previous section. The first step is to make the decomposition

$$\begin{aligned} Y^I &= x^I + iu^I(x, y) , \\ F_I &= y_I + iv_I(x, y) , \end{aligned}$$

and then gather together x^I, y_I to form the special real coordinates $q^a := (x^I, y_I)^T$. Using (6.17) we can write the scalar kinetic term as

$$g_{IJ} \partial_\mu Y^I \partial^\mu \bar{Y}^{\bar{J}} = \left[-\frac{1}{2H} H_{ab} + \frac{1}{4H^2} H_a H_b + \frac{1}{H^2} (\Omega_{ac} q^c)(\Omega_{bd} q^d) \right] \partial_\mu q^a \partial^\mu q^b . \quad (6.33)$$

The Kaluza–Klein scalar can be related to the Hesse potential simply by

$$e^\phi = -i(Y^I \bar{F}_I - F_I \bar{Y}^{\bar{I}}) = -2H ,$$

and so the kinetic term for the KK-scalar can be written as

$$\frac{1}{4} \partial_\mu \phi \partial^\mu \phi = \left[\frac{1}{4H^2} H_a H_b \right] \partial_\mu q^a \partial^\mu q^b . \quad (6.34)$$

We now consider the terms descending from the gauge fields. It is convenient to first define $\hat{q}^a := (\frac{1}{2}\zeta^I, \frac{1}{2}\tilde{\zeta}_I)^T$, which are related to the field strengths by

$$\begin{aligned} \partial_\mu \zeta^I &= \hat{F}_{\mu 0}^I , \\ \partial_\mu \tilde{\zeta}_I &= \hat{G}_{I|\mu 0} . \end{aligned} \quad (6.35)$$

In this case we can now write the terms descending from the gauge fields as

$$\begin{aligned} \frac{1}{2} \epsilon e^{-\phi} \left[\mathcal{I}_{IJ} \partial_\mu \zeta^I \partial^\mu \zeta^J + \mathcal{I}^{IJ} \left(\partial_\mu \tilde{\zeta}_I - \mathcal{R}_{IK} \partial_\mu \zeta^K \right) \left(\partial^\mu \tilde{\zeta}_J - \mathcal{R}_{JL} \partial^\mu \zeta^L \right) \right] \\ = \epsilon \frac{1}{H} \hat{H}_{ab} \partial_\mu \hat{q}^a \partial^\mu \hat{q}^b . \end{aligned}$$

We can then use proposition 6.2 to write this in terms of the Hesse potential

$$\epsilon \frac{1}{H} \hat{H}_{ab} \partial_\mu \hat{q}^a \partial^\mu \hat{q}^b = \epsilon \left[-\frac{1}{2H} H_{ab} + \frac{1}{2H^2} H_a H_b + \frac{2}{H^2} (\Omega_{ac} q^c) (\Omega_{bd} q^d) \right] \partial_\mu \hat{q}^a \partial^\mu \hat{q}^b. \quad (6.36)$$

Using the expressions (6.33), (6.34) and (6.36) we can write the Lagrangian (6.32) in terms of special real coordinates as

$$\begin{aligned} e_3^{-1} \mathcal{L}_3 = & \frac{1}{2} R_3 - \left[-\frac{1}{2H} H_{ab} + \frac{1}{2H^2} H_a H_b \right] \left(\partial_\mu q^a \partial^\mu q^b - \epsilon \partial_\mu \hat{q}^a \partial^\mu \hat{q}^b \right) \\ & - \frac{1}{H^2} \left(q^a \Omega_{ab} \partial_\mu q^b \right)^2 + \epsilon \frac{2}{H^2} \left(q^a \Omega_{ab} \partial_\mu \hat{q}^b \right)^2 \\ & - \frac{1}{4H^2} \left(\partial_\mu \tilde{\phi} + 2\hat{q}^a \Omega_{ab} \partial_\mu \hat{q}^b \right)^2. \end{aligned}$$

It is convenient at this stage to introduce the function \tilde{H} , as defined in equation (6.18), and its second derivatives $\tilde{H}_{ab} = \partial_{a,b}^2 \tilde{H}$. The Lagrangian now takes the simpler form

$$\begin{aligned} e_3^{-1} \mathcal{L}_3 = & \frac{1}{2} R_3 - \tilde{H}_{ab} \left(\partial_\mu q^a \partial^\mu q^b - \epsilon \partial_\mu \hat{q}^a \partial^\mu \hat{q}^b \right) \\ & - \frac{1}{H^2} \left(q^a \Omega_{ab} \partial_\mu q^b \right)^2 + \epsilon \frac{2}{H^2} \left(q^a \Omega_{ab} \partial_\mu \hat{q}^b \right)^2 \\ & - \frac{1}{4H^2} \left(\partial_\mu \tilde{\phi} + 2\hat{q}^a \Omega_{ab} \partial_\mu \hat{q}^b \right)^2. \end{aligned} \quad (6.37)$$

This formula is one of the key results in this thesis, as it provides a new formulation of the c-map in terms of special real coordinates and their corresponding Hesse potential.

Note that the Lagrangian (6.37) still has a U(1) gauge symmetry which descends from overall phase transformations of the X^I coordinates. This is encapsulated by the fact that the ‘metric’ on the target manifold is degenerate along the $J\xi$ direction. In order for the coordinates $q^a, \hat{q}^a, \tilde{\phi}$ to parametrise the physical submanifold we must gauge fix these U(1) transformations by imposing any appropriate constraint. This can only be achieved in a symplectic covariant way by imposing some of the field equations, and so we postpone the discussion of gauge fixing the U(1) until later.

6.3 Real vielbeins

In order to expose the geometry of the c-map target space M described target manifold of the Lagrangian (6.37) we must first revisit the geometry of the projective special Kähler base manifold \bar{N} . We will construct a real orthonormal basis of the co-tangent bundle (i.e. a vielbein basis), and calculate certain properties of the transformation matrices that map a real coordinate basis into a vielbein basis. We also compute the Levi-Civita connection in terms of real coordinates.

We will use these results to construct a real vielbein basis of the co-tangent bundle on the target space M after performing the c-map on \bar{N} . We end by calculating the exterior derivatives of this vielbein basis, which will be used in the next two sections to calculate the Levi-Civita connection in the case of spacelike and timelike reduction respectively.

6.3.1 Real vielbein basis of the PSK base manifold

Let us consider a basis e^M of the co-tangent bundle of a projective special Kähler (PSK) manifold \bar{N} with $\dim_{\mathbb{R}} \bar{N} = 2n$, in which the metric and complex structure are given

by

$$\eta_{MN} = \begin{pmatrix} \mathbb{1}_{k,l} & 0 \\ 0 & \mathbb{1}_{k,l} \end{pmatrix}, \quad J^M_N = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}.$$

where $\mathbb{1}_{k,l} = \begin{pmatrix} -\mathbb{1}_k & 0 \\ 0 & \mathbb{1}_l \end{pmatrix}$ and clearly $k+l = n$. We will call such a basis of one-forms a real vielbein basis. Here we are considering the real formulation of projective special Kähler geometry as presented in section 6.1, so we work with horizontal objects on the corresponding conic affine special Kähler (CASK) manifold. We define the matrices \mathcal{P}_a^M as the projection matrices from a special real coordinate basis to the vielbein basis, i.e.

$$\mathcal{P}_a^M dq^a := e^M.$$

These projection matrices will play an important role in what follows. Note that the tangent space of the CASK manifold has $2n + 2$ dimensions, where as a basis on the PSK manifold only has $2n$ dimensions. Hence the matrices \mathcal{P}_a^M project out two directions, which precisely correspond to the homothety ξ and isometry $J\xi$. This fact is encapsulated in the following lemma:

Lemma 6.1 . The projection matrices \mathcal{P}_a^M satisfy the following properties:

- (i) $\mathcal{P}_a^M q^a = 0$, and $\mathcal{P}_a^M \Omega^{ab} H_b = 0$.
- (ii) $\mathcal{P}_a^M \mathcal{P}_b^N \eta_{MN} = -\frac{1}{2H} H_{ab} + \frac{1}{4H^2} H_a H_b + \frac{1}{H^2} \Omega_{ac} q^c \Omega_{bd} q^d$.
- (iii) $\mathcal{P}_a^M H^{ab} \mathcal{P}_b^N = -\frac{1}{2H} \eta^{MN}$.

Proof. Part (i) follows immediately from the fact that e^M are tangent to the PSK manifold, and so $e^M(\xi) = e^M(J\xi) = 0$. For part (ii) we recall from the discussion in section 6.1.2 that the metric on the PSK manifold can be pulled back to given the following horizontal rank two tensor field on the CASK manifold:

$$H^{(0)} = \left[-\frac{1}{2H} H_{ab} + \frac{1}{4H^2} H_a H_b + \frac{1}{H^2} \Omega_{ac} q^c \Omega_{bd} q^d \right] dq^a \otimes dq^b.$$

By definition the projection matrices satisfy

$$H_{ab}^{(0)} = (\mathcal{P}_a^M \mathcal{P}_b^N \eta_{MN}) dq^a \otimes dq^b,$$

which proves (ii).

For part (iii) we first contract $H_{cb}^{(0)}$ with H^{ac} to find

$$\begin{aligned} H^{ac} H_{cb}^{(0)} &= H^{ac} \left[-\frac{1}{2H} H_{ab} + \frac{1}{4H^2} H_a H_b + \frac{1}{H^2} \Omega_{ac} q^c \Omega_{bd} q^d \right] \\ &= -\frac{1}{2H} \delta_b^a + \frac{1}{4H^2} q^a H_b + \frac{1}{H^2} H^{ac} \Omega_{ce} q^e \Omega_{bd} q^d \\ &= -\frac{1}{2H} \delta_b^a + \frac{1}{4H^2} q^a H_b - \frac{1}{4H^2} \Omega^{ac} H_c \Omega_{bd} q^d, \end{aligned}$$

where in the last line we used the identity (6.11a). Contracting this with \mathcal{P}_a^M we find

$$\mathcal{P}_a^M H^{ac} H_{cb}^{(0)} = -\frac{1}{2H} \mathcal{P}_b^M,$$

where we have used part (i). Next we can use part (ii) to write this as

$$\mathcal{P}_a^M H^{ac} \mathcal{P}_c^N \eta_{NP} \mathcal{P}_b^P = -\frac{1}{2H} \eta^{MN} \eta_{NP} \mathcal{P}_b^P ,$$

and since $\eta_{NP} \mathcal{P}_b^P$ is of rank n we can remove it from both sides to get

$$\mathcal{P}_a^M H^{ac} \mathcal{P}_c^N = -\frac{1}{2H} \eta^{MN} .$$

□

The projection π from the CASK manifold N to the PSK manifold \bar{N} is holomorphic, and so by remark 2.2 the complex structures are related by $d\pi \circ J_{\text{CASK}} = J_{\text{PSK}} \circ d\pi$, which we can write in components as

$$J_N^M \mathcal{P}_a^N = \mathcal{P}_b^M J_a^b .$$

It follows that the complex structure on N can be written as

$$J_P^M = H \mathcal{P}^M \Omega^{-1} \mathcal{P}^N \eta_{NP} . \quad (6.38)$$

It is straightforward to check that this indeed gives $J^2 = -\text{Id}_{TM}$.

The Levi-Civita connection one-form ω_N^M on the PSK manifold \bar{N} is uniquely defined by being metric compatible and torsion-free. Recall from proposition 2.1 that the metric compatibility condition is satisfied iff

$$\omega_N^M \in \mathfrak{so}(2k, 2l) \quad \Rightarrow \quad \omega^{MN} = \omega_Q^M \eta^{QN} = -\omega^{NM} ,$$

while the torsion-free condition is given by

$$de^M = -\omega_N^M \wedge e^N .$$

After performing the calculation, the Levi-Civita connection one-form is found to be

$$\omega_P^M = \left(-(dq\Omega q) \mathcal{P}^M \Omega^{-1} \mathcal{P}^N + H d\mathcal{P}_a^M H^{ab} \mathcal{P}_b^N - H \mathcal{P}_a^M H^{ab} d\mathcal{P}_b^N \right) \eta_{NP} .$$

The Levi-Civita connection and the complex structure satisfy the identity

$$J_N^M \omega_P^N = \omega_N^M J_P^N , \quad (6.39)$$

which will be particularly important later when transforming between real and complex coordinates.

6.3.2 Real vielbein basis after performing the c-map

We now turn our attention to the target manifold M after performing the c-map with base space \bar{N} . From the Lagrangian (6.37) we can read off the metric on M :

$$\begin{aligned} g = & \tilde{H}_{ab} \left(dq^a \otimes dq^b - \epsilon d\hat{q}^a \otimes d\hat{q}^b \right) \\ & + \frac{1}{H^2} (q\Omega dq) \otimes (q\Omega dq) - \epsilon \frac{2}{H^2} (q\Omega d\hat{q}) \otimes (q\Omega d\hat{q}) \\ & + \frac{1}{4H^2} \left(d\tilde{\phi} + 2\hat{q}\Omega d\hat{q} \right) \otimes \left(d\tilde{\phi} + 2\hat{q}\Omega d\hat{q} \right) . \end{aligned}$$

To expose the geometry it is more convenient to write the metric in terms of $H_{ab}^{(0)}$ rather than \tilde{H}_{ab}

$$\begin{aligned} g = & H_{ab}^{(0)} dq^a \otimes dq^b + \frac{1}{4H^2} \left(H_a dq^a \right) \otimes \left(H_b dq^b \right) \\ & - \epsilon \left[H_{ab}^{(0)} d\hat{q}^a \otimes d\hat{q}^b + \frac{1}{4H^2} \left(H_a d\hat{q}^a \right) \otimes \left(H_b d\hat{q}^b \right) + \frac{1}{H^2} (q\Omega d\hat{q}) \otimes (q\Omega d\hat{q}) \right] \\ & + \frac{1}{4H^2} \left(d\tilde{\phi} + 2\hat{q}\Omega d\hat{q} \right) \otimes \left(d\tilde{\phi} + 2\hat{q}\Omega d\hat{q} \right) . \end{aligned}$$

Let us define the following vielbein basis of the co-tangent bundle of M :

$$\begin{aligned} e^M &:= \mathcal{P}_a^M dq^a , & \hat{e}^M &:= -\mathcal{P}_a^M d\hat{q}^a , \\ u &:= \frac{1}{2H} H_a dq^a , & \hat{u} &:= \frac{1}{2H} H_a d\hat{q}^a , \\ w &:= \frac{1}{2H} \left(d\tilde{\phi} + 2\hat{q}\Omega d\hat{q} \right) , & \hat{v} &:= \frac{1}{H} d\hat{q}\Omega q . \end{aligned}$$

The metric can then be written in terms of these vielbeins as

$$g = e^M \otimes e^N \eta_{MN} + u \otimes u + w \otimes w - \epsilon \left(\hat{e}^M \otimes \hat{e}^N \eta_{MN} + \hat{u} \otimes \hat{u} + \hat{v} \otimes \hat{v} \right) .$$

It is also useful to define the one-form v in analogy with \hat{v} , i.e.

$$v := \frac{1}{H} dq\Omega q ,$$

however this one-form does not appear in the expression for metric.

Lemma 6.2 . The vector fields corresponding to the vielbein one-forms are given by

$$\begin{aligned} (e^M)^{-1} &= -2H\mathcal{P}_a^M H^{ab} \frac{\partial}{\partial q^b} , \\ u^{-1} &= q^a \frac{\partial}{\partial q^a} , \\ w^{-1} &= 2H \frac{\partial}{\partial \tilde{\phi}} , \\ (\hat{e}^M)^{-1} &= 2H\mathcal{P}_a^M H^{ab} \frac{\partial}{\partial \hat{q}^b} + 4H\mathcal{P}_a^M H^{ab} \Omega_{bc} \hat{q}^c \frac{\partial}{\partial \tilde{\phi}} , \\ \hat{u}^{-1} &= q^a \frac{\partial}{\partial \hat{q}^a} - 2\hat{q}^a \Omega_{ab} q^b \frac{\partial}{\partial \tilde{\phi}} , \\ \hat{v}^{-1} &= \frac{1}{2} H_a \Omega^{ab} \frac{\partial}{\partial \hat{q}^b} + H_a \hat{q}^a \frac{\partial}{\partial \tilde{\phi}} . \end{aligned}$$

Proof. Direct calculation. Make use of lemma 6.1. □

We would now like to take the exterior derivative of the elements of the vielbein basis. First, let us define the functions

$$H_{MNP} := \left(-2H\mathcal{P}_a^Q H^{ad} \eta_{QM} \right) \left(-2H\mathcal{P}_b^R H^{be} \eta_{RN} \right) \left(-2H\mathcal{P}_c^S H^{cf} \eta_{SP} \right) H_{def} .$$

Lemma 6.3 . The rank three tensor field H_{MNP} satisfies the following identities:

- (i) $\mathcal{P}_a^M \mathcal{P}_b^N \mathcal{P}_c^P H_{MNP} = H_{abc}$,
- (ii) $H(\mathcal{P}_a^M dH^{ab} \mathcal{P}_b^N) \eta_{NP} = -\frac{1}{4H} \eta^{MQ} H_{QNP} e^N$,
- (iii) $H_{MNP} J_R^N J_S^P = H_{MRS}$.

Proof. Direct calculation. Make use of lemma 6.1 and identities (6.8) and (6.11). \square

We are now in a position to calculate the exterior derivatives of the basis of real vielbein one-forms

$$\begin{aligned}
de^M &= -\omega^M_P \wedge e^P, \\
du &= 0, \\
dw &= 2w \wedge u + 2\hat{v} \wedge \hat{u} + J^N_P \hat{e}^P \wedge \hat{e}^Q \eta_{NQ}, \\
d\hat{u} &= \hat{u} \wedge u + v \wedge \hat{v} + e^N \wedge \hat{e}^Q \eta_{NQ}, \\
d\hat{v} &= \hat{v} \wedge u + \hat{u} \wedge v + J^N_P e^P \wedge \hat{e}^Q \eta_{NQ}, \\
d\hat{e}^M &= \hat{e}^M \wedge u + e^M \wedge \hat{u} + J^M_N e^N \wedge \hat{v} + J^M_N \hat{e}^N \wedge v \\
&\quad - \omega^M_P \wedge \hat{e}^P - \frac{1}{4H} \eta^{MQ} H_{QNP} e^N \wedge \hat{e}^P.
\end{aligned}$$

From these simple expression we will be able to recover the celebrated result of Ferrara and Sabharwal, while going on to prove that the timelike c-map is para-quaternion Kähler. We will also prove the existence of two metric compatible integrable complex structure in the image of both the spacelike and timelike c-map.

6.4 Quaternion Kähler structure of spacelike reduction

In this section we will specialise to the spacelike c-map, in which case we must make the choice $\epsilon = -1$ in the calculations of sections 6.2 and 6.3.

In [26] the following complex vielbein basis of the complexified co-tangent bundle is defined:

$$\begin{aligned}
u^{\mathbb{C}} &:= ie^{-\phi/2} \left(X^I d\tilde{\zeta}_I - F_I d\zeta^I \right), \\
v^{\mathbb{C}} &:= e^{-\phi} \left[\frac{1}{2} de^\phi + i \left(d\tilde{\phi} + \frac{1}{2} (\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I) \right) \right], \\
E^{\mathbb{C}A} &:= -ie^{-\phi/2} P_L^A N^{LI} \left(d\tilde{\zeta}_I - \mathcal{N}_{IJ} d\zeta^J \right), \\
e^{\mathbb{C}A} &:= P_I^A dX^I = e_i^A dz^i.
\end{aligned}$$

We can write the complex one-forms in terms of the real vielbein one-forms, introduced in the previous section, as:

$$\begin{aligned}
u^{\mathbb{C}} &= ie^{-\phi} \left(Y^I d\tilde{\zeta}_I - F_I d\zeta^I \right) \\
&= -i \frac{1}{2H} \left[x^I d\tilde{\zeta}_I - y_I d\zeta^I + i \left(u^I d\tilde{\zeta}_I - v_I d\zeta^I \right) \right] \\
&= -i \frac{1}{2H} \left[2q^a \Omega_{ab} d\hat{q}^b + i (H_a d\hat{q}^a) \right] = \hat{u} + i\hat{v}, \\
v^{\mathbb{C}} &= -\frac{1}{2H} \left[d(-H) + i \left(d\tilde{\phi} + \frac{1}{2} (\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I) \right) \right] = u + iw.
\end{aligned}$$

$$\begin{aligned}
E^{\mathbb{C}A} &= -2i(\delta_M^A + iJ_M^A)\mathcal{P}_a{}^M H^{ab}\Omega_{bc}\hat{q}^c \\
&= -i(\delta_M^A + iJ_M^A)\mathcal{P}_a{}^M J_b^a \hat{q}^b \\
&= -i(\delta_M^A + iJ_M^A)J_N^M \mathcal{P}_a{}^N \hat{q}^a = (\delta_M^A + iJ_M^A)\hat{e}^M, \\
e^{\mathbb{C}A} &= (\delta_M^A + iJ_M^A)e^M.
\end{aligned}$$

Gathering these together, we can write the complex vielbein basis in terms of the real vielbein basis simply as

$$\begin{aligned}
u^{\mathbb{C}} &= \hat{u} + i\hat{v}, & e^{\mathbb{C}A} &= e^A + iJ_N^A e^N, \\
v^{\mathbb{C}} &= u + iw, & E^{\mathbb{C}A} &= \hat{e}^A + iJ_N^A \hat{e}^N.
\end{aligned}$$

We would like to take the exterior derivatives of the complex vielbeins. Although these have already been calculated using complex vielbeins in [26], we will choose to calculate them independently using real vielbeins. If our calculations are correct then we should recover the results of [26]. Firstly, we can write the derivative of $u^{\mathbb{C}}$ as

$$\begin{aligned}
du^{\mathbb{C}} &= \hat{u} \wedge u + v \wedge \hat{v} + i\hat{v} \wedge u + i\hat{u} \wedge v + \hat{e}^N \wedge e^N + iJ_P^N e^P \wedge \hat{e}^N \\
&= (-u - iv) \wedge u^{\mathbb{C}} - \bar{E}^{\mathbb{C}A} \wedge e^{\mathbb{C}A}.
\end{aligned}$$

Since

$$-iv = \frac{\bar{X}NdX - XNd\bar{X}}{2\bar{X}NX},$$

this is in agreement with [26]. The derivative of $v^{\mathbb{C}}$ is given by

$$\begin{aligned}
dv^{\mathbb{C}} &= 2iw \wedge u + 2i\hat{v} \wedge \hat{u} + iJ_P^N \hat{e}^P \wedge \hat{e}^N \\
&= u^{\mathbb{C}} \wedge \bar{u}^{\mathbb{C}} + v^{\mathbb{C}} \wedge \bar{v}^{\mathbb{C}} + E^{\mathbb{C}A} \wedge \bar{E}^{\mathbb{C}A},
\end{aligned}$$

which is again in agreement with [26]. The derivative of $e^{\mathbb{C}A}$ is given by

$$de^{\mathbb{C}A} = -(\delta_M^A + iJ_M^A)\omega_N^M \wedge e^N.$$

Comparing this to the complex connection

$$de^{\mathbb{C}A} = -\omega^{\mathbb{C}A}_B \wedge e^{\mathbb{C}B},$$

we find the standard relation between the real and complex connections:

$$\omega^{\mathbb{C}A}_B = \frac{1}{2}(\delta_M^A + iJ_M^A)\omega_N^M(\delta_B^N - iJ_B^N).$$

Finally, taking derivatives of $E^{\mathbb{C}A}$ we find

$$\begin{aligned}
dE^{\mathbb{C}A} &= (-u + iv) \wedge E^{\mathbb{C}A} - \bar{u}^{\mathbb{C}} \wedge e^{\mathbb{C}A} - \omega^{\mathbb{C}A}_B \wedge E^{\mathbb{C}B} \\
&\quad + \frac{i}{2\bar{X}NX} \delta^{AD} \bar{F}_{DBC} \bar{e}^{\mathbb{C}B} \wedge \bar{E}^{\mathbb{C}C}.
\end{aligned}$$

This is again in agreement with [26]. In this calculation we used the fact that

$$\begin{aligned}
(\delta_A^M + iJ_A^M)(\delta_B^N + iJ_B^N)(\delta_C^P + iJ_C^P)H_{MNP} &= i\bar{F}_{ABC} \\
\Rightarrow (\delta_A^M + iJ_A^M)H_{MBP}\hat{e}^P &= i\bar{F}_{ABC}\bar{E}^{\mathbb{C}C}.
\end{aligned}$$

Gathering everything together we can write the derivatives of the complex vielbeins as

$$\begin{aligned} du^{\mathbb{C}} &= (-u - iv) \wedge u^{\mathbb{C}} - \bar{E}^{\mathbb{C}A} \wedge e^{\mathbb{C}A}, \\ dv^{\mathbb{C}} &= u^{\mathbb{C}} \wedge \bar{u}^{\mathbb{C}} + v^{\mathbb{C}} \wedge \bar{v}^{\mathbb{C}} + E^{\mathbb{C}A} \wedge \bar{E}^{\mathbb{C}A}, \\ de^{\mathbb{C}A} &= -\omega^{\mathbb{C}A}_B \wedge e^{\mathbb{C}B}, \\ dE^{\mathbb{C}A} &= (-u + iv) \wedge E^{\mathbb{C}A} - \bar{u}^{\mathbb{C}} \wedge e^{\mathbb{C}A} - \omega^{\mathbb{C}A}_B \wedge E^{\mathbb{C}B} \\ &\quad + \frac{i}{2\bar{X}NX} \delta^{AD} \bar{F}_{DBC} \bar{e}^{\mathbb{C}B} \wedge \bar{E}^{\mathbb{C}C}, \end{aligned}$$

which are in precise agreement with [26].

It is useful to gather together the complex vielbein one-forms into a $2 \times (2n + 2)$ matrix

$$\mathcal{U}^{\mathcal{A}m} := \frac{1}{\sqrt{2}} \begin{pmatrix} u^{\mathbb{C}} & e^{\mathbb{C}A} & \bar{v}^{\mathbb{C}} & \bar{E}^{\mathbb{C}A} \\ v^{\mathbb{C}} & E^{\mathbb{C}A} & -\bar{u}^{\mathbb{C}} & -\bar{e}^{\mathbb{C}A} \end{pmatrix}.$$

The metric can then be written as $g = \varepsilon_{\mathcal{A}\mathcal{B}} \rho_{mn} \mathcal{U}^{\mathcal{A}m} \otimes \mathcal{U}^{\mathcal{B}n}$, where

$$\varepsilon_{\mathcal{A}\mathcal{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho_{mn} = \begin{pmatrix} 0 & \mathbb{1}_{n+1} \\ -\mathbb{1}_{n+1} & 0 \end{pmatrix}.$$

The matrix $\mathcal{U}^{\mathcal{A}m}$ can be written as $\mathcal{U}^{\mathcal{A}m} = h^{\mathcal{A}} \otimes \mathcal{E}^m$ where

$$h^{\mathcal{A}} = \frac{1}{\sqrt{2}} (\mathbb{1} + i\mathbb{I}, \mathbb{K} + i\mathbb{J}), \quad \mathcal{E}^m = (e_1, \dots, e_{n+1}, \mathbb{K}e_1, \dots, \mathbb{K}e_{n+1}),$$

$$\begin{aligned} e_1, \dots, e_{n+1} &= \hat{u}, e^A \\ \mathbb{I}e_1, \dots, \mathbb{I}e_{n+1} &= \hat{v}, J^A_M e^M \\ \mathbb{J}e_1, \dots, \mathbb{J}e_{n+1} &= w, J^A_M \hat{e}^M \\ \mathbb{K}e_1, \dots, \mathbb{K}e_{n+1} &= u, \hat{e}^A. \end{aligned}$$

The endomorphisms $\mathbb{I}, \mathbb{J}, \mathbb{K}$ can then be completely determined by the above expressions and the requirement that satisfy the quaternion relations $\mathbb{I}^2 = \mathbb{J}^2 = \mathbb{K}^2 = -1$ and $\mathbb{I}\mathbb{J} = \mathbb{K}$.

The Levi-Civita connection one-form Ω is uniquely defined through the expression

$$d\mathcal{U}^{\mathcal{A}m} + \Omega^{\mathcal{A}m}_{\mathcal{B}n} \wedge \mathcal{U}^{\mathcal{B}n} = 0,$$

and the requirement that $\Omega \in \mathfrak{so}(4k, 4l + 4)$. For any quaternion Kähler manifold one can decompose the Levi-Civita one-form as

$$\Omega = p \otimes \mathbb{1}_{2n+2} + \mathbb{1}_2 \otimes \begin{pmatrix} q & t \\ -\bar{t} & \bar{q} \end{pmatrix}. \quad (6.40)$$

The matrices p, q, t satisfy the following properties:

$$\begin{aligned} \text{Tr}(p) &= 0, & q^\dagger &= -q, \\ p^\dagger &= -p, & t^T &= t, \end{aligned}$$

which imply that

$$p \in \mathfrak{sp}(1), \quad \begin{pmatrix} q & t \\ -\bar{t} & \bar{q} \end{pmatrix} \in \mathfrak{sp}(n + 1).$$

One can determine the matrices p, q, t through the expression

$$de^{AI} + P^A_{\mathcal{B}} e^{\mathcal{B}I} + q^I_J e^{AJ} + t^I_J \varepsilon^{\mathcal{A}\mathcal{B}} \bar{e}^{\mathcal{B}J} = 0, \quad (6.41)$$

where

$$e^{AI} = \begin{pmatrix} u^{\mathbb{C}} & v^{\mathbb{C}} \\ e^{\mathbb{C}A} & E^{\mathbb{C}A} \end{pmatrix}, \quad \varepsilon^{\mathcal{A}\mathcal{B}} \bar{e}^{\mathcal{B}J} = \begin{pmatrix} \bar{v}^{\mathbb{C}} & -\bar{u}^{\mathbb{C}} \\ \bar{E}^{\mathbb{C}A} & -\bar{e}^{\mathbb{C}A} \end{pmatrix}.$$

Theorem 6.1 (Ferrara–Sabharwal). The target manifold in the image of the space-like c-map is quaternion Kähler.

Proof. The Levi-Civita connection takes the form (6.40), where the matrices p, q, t are given by [26]

$$p = \begin{pmatrix} \frac{1}{4}(v^{\mathbb{C}} - \bar{v}^{\mathbb{C}}) - \frac{1}{4} \frac{\bar{X}NdX - d\bar{X}NX}{\bar{X}NX} & -u^{\mathbb{C}} \\ \bar{u}^{\mathbb{C}} & -\frac{1}{4}(v^{\mathbb{C}} - \bar{v}^{\mathbb{C}}) + \frac{1}{4} \frac{\bar{X}NdX - d\bar{X}NX}{\bar{X}NX} \end{pmatrix},$$

$$q = \begin{pmatrix} -\frac{3}{4}(v^{\mathbb{C}} - \bar{v}^{\mathbb{C}}) - \frac{1}{4} \frac{\bar{X}NdX - d\bar{X}NX}{\bar{X}NX} & \bar{E}^{\mathbb{C}A} \\ -E^{\mathbb{C}A} & \omega^{\mathbb{C}A}_B - \frac{1}{4}(v^{\mathbb{C}} - \bar{v}^{\mathbb{C}})\delta^A_B + \frac{1}{4} \frac{\bar{X}NdX - d\bar{X}NX}{\bar{X}NX} \delta^A_B \end{pmatrix},$$

$$t = \begin{pmatrix} 0 & 0 \\ 0 & \frac{i}{2\bar{X}NX} \delta^{AD} \bar{F}_{DBC} \bar{E}^{\mathbb{C}C} \end{pmatrix}.$$

Since $\text{Tr}(p) = 0$, $q^\dagger = -q$, $p^\dagger = -p$, and $t^T = t$ it follows that

$$p \in \mathfrak{sp}(1), \quad \begin{pmatrix} q & t \\ -\bar{t} & \bar{q} \end{pmatrix} \in \mathfrak{sp}(n+1),$$

and, hence, the Levi-Civita connection one-form Ω takes values in the Lie algebra $\mathfrak{sp}(n+1) \cdot \mathfrak{sp}(1)$. By corollary 2.1 the Riemann holonomy group is given by $\text{Hol}(D) \subset \text{Sp}(n+1) \cdot \text{Sp}(1)$ and the manifold is quaternion Kähler. \square

We can write these in terms of real vielbeins as

$$p = \begin{pmatrix} \frac{i}{2}w + \frac{i}{2}v & -\hat{u} - i\hat{v} \\ \hat{u} - i\hat{v} & -\frac{i}{2}w - \frac{i}{2}v \end{pmatrix},$$

$$q = \begin{pmatrix} -\frac{3i}{2}w + \frac{i}{2}v & (\delta^A_M - iJ^A_M) \hat{e}^M \\ -(\delta^A_M + iJ^A_M) \hat{e}^M & (\delta^A_M + iJ^A_M) \omega^M_B - \frac{i}{2}(w + v) \delta^A_B \end{pmatrix},$$

$$t = \begin{pmatrix} 0 & 0 \\ 0 & -(\delta^{AM} + iJ^{AM}) \frac{1}{4H} H_{MBP} \hat{e}^P \end{pmatrix}.$$

The complex structure \mathbb{I} acts on the complex one-forms by

$$\begin{aligned}\mathbb{I}(u^{\mathbb{C}}) &= iu^{\mathbb{C}}, \\ \mathbb{I}(v^{\mathbb{C}}) &= -iv^{\mathbb{C}}, \\ \mathbb{I}(e^{\mathbb{C}A}) &= ie^{\mathbb{C}A}, \\ \mathbb{I}(E^{\mathbb{C}A}) &= -iE^{\mathbb{C}A}.\end{aligned}$$

Consider also a second complex structure, \mathbb{I}' , defined by

$$\begin{aligned}\mathbb{I}'(u^{\mathbb{C}}) &= iu^{\mathbb{C}}, \\ \mathbb{I}'(v^{\mathbb{C}}) &= iv^{\mathbb{C}}, \\ \mathbb{I}'(e^{\mathbb{C}A}) &= ie^{\mathbb{C}A}, \\ \mathbb{I}'(E^{\mathbb{C}A}) &= -iE^{\mathbb{C}A}.\end{aligned}$$

Note that the only difference between the two complex structures is in the action of $v^{\mathbb{C}}$. The complex structure \mathbb{I} is part of the quaternionic structure, but \mathbb{I}' is not.

Theorem 6.2 . The complex structures \mathbb{I} and \mathbb{I}' are integrable.

Proof. Consider first \mathbb{I} . A basis of $T^{*(1,0)}M$ is given by $\{u^{\mathbb{C}}, \bar{v}^{\mathbb{C}}, e^{\mathbb{C}A}, \bar{E}^{\mathbb{C}A}\}$. Since no $(0,2)$ -form appears in the exterior derivatives of these basis vectors the distribution is integrable by the Newlander-Nirenberg theorem [35].

A basis of $T^{*(1,0)}M$ for \mathbb{I}' is given by $\{u^{\mathbb{C}}, v^{\mathbb{C}}, e^{\mathbb{C}A}, \bar{E}^{\mathbb{C}A}\}$. Again by the Newlander-Nirenberg theorem it is an integrable distribution. \square

6.5 Para-quaternion Kähler structure of timelike reduction

In this final section on the c-map we present the proof that the target manifold in the image of the timelike c-map is para-quaternion Kähler by explicitly computing the Levi-Civita connection one form.

In order to gain some intuition from the calculations of Ferrara and Sabharwal in the spacelike case, we will first make a symplectic rotation of the vielbein basis so that it is in a form more useful for comparison with timelike reduction.

6.5.1 Symplectic rotation of Ferrara–Sabharwal basis

We will now make the particular symplectic rotation of the Ferrara–Sabharwal basis given by the symplectic matrices

$$T = \frac{1}{(1+i)} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}, \quad S = \frac{1}{(1+i)} \left(\begin{array}{ccc|ccc} i & & & i & & \\ & -i & & & i & \\ & & -i & & & i \\ \hline -1 & & & 1 & & \\ & -1 & & & -1 & \\ & & -1 & & & -1 \\ & & & -1 & & \end{array} \right).$$

Under such a rotation the vielbeins transform as

$$\mathcal{U}^{Am} \longrightarrow \mathcal{U}'^{Am} = \frac{1}{\sqrt{2}} \begin{pmatrix} \text{Im}(v^{\mathbb{C}}) + i\text{Re}(u^{\mathbb{C}}) & \text{Im}(e^{\mathbb{C}}) + i\text{Re}(E^{\mathbb{C}}) & \text{Re}(v^{\mathbb{C}}) - i\text{Im}(u^{\mathbb{C}}) & -\text{Re}(e^{\mathbb{C}}) + i\text{Im}(E^{\mathbb{C}}) \\ \text{Re}(v^{\mathbb{C}}) + i\text{Im}(u^{\mathbb{C}}) & -\text{Re}(e^{\mathbb{C}}) - i\text{Im}(E^{\mathbb{C}}) & -\text{Im}(v^{\mathbb{C}}) + i\text{Re}(u^{\mathbb{C}}) & -\text{Im}(e^{\mathbb{C}}) + i\text{Re}(E^{\mathbb{C}}) \end{pmatrix}.$$

One can write this as a transformation of the individual vielbein one-forms, which transform as

$$\begin{aligned} u^{\mathbb{C}} &\longrightarrow u'^{\mathbb{C}} = \text{Im}(v^{\mathbb{C}}) + i\text{Re}(u^{\mathbb{C}}) = w + i\hat{u}, \\ v^{\mathbb{C}} &\longrightarrow v'^{\mathbb{C}} = \text{Re}(v^{\mathbb{C}}) + i\text{Im}(u^{\mathbb{C}}) = u + i\hat{v}, \\ e^{\mathbb{C}A} &\longrightarrow e'^{\mathbb{C}A} = \text{Im}(e^{\mathbb{C}A}) + i\text{Re}(E^{\mathbb{C}A}) = J^A_M e^M + i\hat{e}^A, \\ E^{\mathbb{C}A} &\longrightarrow E'^{\mathbb{C}A} = -\text{Re}(e^{\mathbb{C}A}) - i\text{Im}(E^{\mathbb{C}A}) = -e^A - iJ^A_M \hat{e}^M. \end{aligned}$$

The benefit of making this rotation is that the vielbein basis is now better adapted to the timelike version of the c-map.

Before the rotation the Levi-Civita connection one-form Ω is defined through the expression

$$d\mathcal{U}^{Am} + \Omega^{Am}_{Bn} \wedge \mathcal{U}^{Bn} = 0.$$

After the rotation it is given by a new Ω' defined by

$$d\mathcal{U}'^{Am} + \Omega'^{Am}_{Bn} \wedge \mathcal{U}'^{Bn} = 0.$$

The two are related by the expression

$$\Omega'^{Am}_{Bn} = T^A_B S^m_n \Omega^{Bn}_{Cr} (T^{-1})^C_D (S^{-1})^r_s.$$

Let us write the 2×2 matrix p as

$$p = \begin{pmatrix} x & y \\ -\bar{y} & -x \end{pmatrix},$$

where x is purely imaginary. The matrix p satisfies $\text{Tr}(p) = 0$ and $p^\dagger = -p$. First, note that

$$T^{-1} = \frac{(1+i)}{2} \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}.$$

We can then write

$$p' := TpT^{-1} = \begin{pmatrix} -i\text{Im}(y) & ix + i\text{Re}(y) \\ -ix + i\text{Re}(y) & i\text{Im}(y) \end{pmatrix}.$$

The matrix p' satisfies $\text{Tr}(p') = 0$ and $p'^\dagger = -p'$ which implies that $p' \in \mathfrak{sp}(1)$.

Let us write the $(n+1) \times (n+1)$ matrices q and t as

$$q = \left(\begin{array}{c|c} a & \alpha \\ \hline -\bar{\alpha} & A \end{array} \right), \quad t = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & B \end{array} \right),$$

where a is purely imaginary, $A^\dagger = -A$ and $B^T = B$. The matrices q and t satisfy $q^\dagger = -q$ and $t^T = t$. The inverse of S is given by

$$S^{-1} = \frac{(1+i)}{2} \left(\begin{array}{ccc|ccc} -i & & & -1 & & \\ & i & & & -1 & \\ & & i & & & -1 \\ \hline -i & & & 1 & & \\ & -i & & & -1 & \\ & & -i & & & -1 \\ & & & -i & & \end{array} \right).$$

Under the transformation

$$S \begin{pmatrix} q & t \\ -\bar{t} & \bar{q} \end{pmatrix} S^{-1},$$

the matrices q and t transform as

$$q \longrightarrow q' = \left(\begin{array}{c|c} 0 & -i\text{Im}(\alpha) \\ \hline -i\text{Im}(\alpha) & \text{Re}(A) - i\text{Im}(B) \end{array} \right),$$

$$t \longrightarrow t' = \left(\begin{array}{c|c} \text{Im}(a) & -i\text{Re}(\alpha) \\ \hline -i\text{Re}(\alpha) & -\text{Im}(A) + i\text{Re}(B) \end{array} \right).$$

One can easily show that the matrices q' and t' satisfy $q'^\dagger = -q'$ and $t'^T = t'$.

We can now plug in the values of x, y, a, α, A, B , found in the previous section, to obtain the connection in the rotated basis. These are given by

$$p' = \begin{pmatrix} i\hat{v} & -\frac{1}{2}(w+v) - i\hat{u} \\ \frac{1}{2}(w+v) - i\hat{u} & -i\hat{v} \end{pmatrix},$$

$$q' = \begin{pmatrix} 0 & iJ^A_M \hat{e}^M \\ iJ^A_M \hat{e}^M & \omega^A_B + \frac{i}{4H} J^{AM} H_{MBP} \hat{e}^P \end{pmatrix},$$

$$t' = \begin{pmatrix} -\frac{3}{2}w + \frac{1}{2}v & -i\hat{e}^A \\ -i\hat{e}^A & -J^A_M \omega^M_B - \frac{i}{4H} \delta^{AM} H_{MBP} \hat{e}^P + \frac{1}{2}(w+v)\delta^A_B \end{pmatrix}.$$

6.5.2 Para-quaternion Kähler structure

We now specialise to the reduction over time, so we must make the choice $\epsilon = 1$ in the calculations of sections 6.2 and 6.3.

Let us define the following real basis of the co-tangent bundle of the target manifold in the image of the timelike c-map:

$$\begin{aligned} u^{\mathbb{R}} &:= -w + \hat{u} , & \bar{u}^{\mathbb{R}} &:= -w - \hat{u} , \\ v^{\mathbb{R}} &:= u + \hat{v} , & \bar{v}^{\mathbb{R}} &:= u - \hat{v} , \\ e^{\mathbb{R}A} &:= J^A_M e^M + \hat{e}^A , & \bar{e}^{\mathbb{R}A} &:= J^A_M e^M - \hat{e}^A , \\ E^{\mathbb{R}A} &:= -e^A - J^A_M \hat{e}^M , & \bar{E}^{\mathbb{R}A} &:= -e^A + J^A_M \hat{e}^M . \end{aligned}$$

We can collect these into a $4n + 4$ component vielbein

$$\mathcal{U}^{\mathcal{A}m} := \frac{1}{\sqrt{2}} \begin{pmatrix} u^{\mathbb{R}} & e^{\mathbb{R}A} & \bar{v}^{\mathbb{R}} & \bar{E}^{\mathbb{R}A} \\ v^{\mathbb{R}} & E^{\mathbb{R}A} & -\bar{u}^{\mathbb{R}} & -\bar{e}^{\mathbb{R}A} \end{pmatrix} .$$

The metric can then be written as $g = \varepsilon_{\mathcal{A}\mathcal{B}} \rho_{mn} \mathcal{U}^{\mathcal{A}m} \otimes \mathcal{U}^{\mathcal{B}n}$, where

$$\varepsilon_{\mathcal{A}\mathcal{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \quad \rho_{mn} = \begin{pmatrix} 0 & \mathbb{1}_{n+1} \\ -\mathbb{1}_{n+1} & 0 \end{pmatrix} .$$

The vielbein can be written as $\mathcal{U}^{\mathcal{A}m} = h^{\mathcal{A}} \otimes \mathcal{E}^m$, where

$$\begin{aligned} h^{\mathcal{A}} &= \frac{1}{\sqrt{2}} \left(\mathbb{1} + \tilde{\mathbb{I}}, \tilde{\mathbb{K}} + \tilde{\mathbb{J}} \right) , & \mathcal{E}^m &= (e_1, \dots, e_{n+1}, \tilde{\mathbb{K}}e_1, \dots, \tilde{\mathbb{K}}e_{n+1}) , \\ e_1, \dots, e_{n+1} &= -w, J^A_M e^M \\ \tilde{\mathbb{I}}e_1, \dots, \tilde{\mathbb{I}}e_{n+1} &= \hat{u}, \hat{e}^A \\ \tilde{\mathbb{J}}e_1, \dots, \tilde{\mathbb{J}}e_{n+1} &= \hat{v}, -J^A_M \hat{e}^M \\ \tilde{\mathbb{K}}e_1, \dots, \tilde{\mathbb{K}}e_{n+1} &= u, -e^A . \end{aligned}$$

The endomorphisms $\tilde{\mathbb{I}}, \tilde{\mathbb{J}}, \tilde{\mathbb{K}}$ can be completely determined by requiring that they satisfy the quaternion relations $\tilde{\mathbb{I}}^2 = \tilde{\mathbb{J}}^2 = 1$, $\tilde{\mathbb{K}}^2 = -1$ and $\tilde{\mathbb{I}}\tilde{\mathbb{J}} = \tilde{\mathbb{K}}$.

The Levi-Civita connection one-form Ω is uniquely defined through the requirement that it is torsion-free

$$d\mathcal{U}^{\mathcal{A}m} + \Omega^{\mathcal{A}m}_{\mathcal{B}n} \wedge \mathcal{U}^{\mathcal{B}n} = 0 ,$$

and that it is metric compatible, i.e. $\Omega \in \mathfrak{so}(2n+2, 2n+2)$. For any para-QK manifold one can decompose the Levi-Civita one-form into

$$\Omega = p \otimes \mathbb{1}_{2n+2} + \mathbb{1}_2 \otimes \begin{pmatrix} q & t \\ -t^T & -q^T \end{pmatrix} . \quad (6.42)$$

The matrices p, q, t satisfy the following properties:

$$\begin{aligned} \text{Tr}(p) = 0 & \Rightarrow p \in \mathfrak{sl}(2, \mathbb{R}) \approx \mathfrak{sp}(2, \mathbb{R}) , \\ t = t^T & \Rightarrow \begin{pmatrix} q & t \\ -t^T & -q^T \end{pmatrix} \in \mathfrak{sp}(2n+2, \mathbb{R}) . \end{aligned}$$

Hence $\Omega \in \mathfrak{sp}(2, \mathbb{R}).\mathfrak{sp}(2n+2, \mathbb{R}) \subseteq \mathfrak{so}(2n+2, 2n+2)$.

Theorem 6.3 . The target manifold in the image of the timelike c-map is para-quaternion Kähler.

Proof. The Levi-Civita connection one-form decomposes according to (6.42), where p, q, t are given by

$$p = \begin{pmatrix} \hat{v} & -\frac{1}{2}(-w+v) - \hat{u} \\ \frac{1}{2}(-w+v) - \hat{u} & -\hat{v} \end{pmatrix},$$

$$q = \begin{pmatrix} 0 & J^A_M \hat{e}^M \\ J^A_M \hat{e}^M & \omega^A_B + \frac{1}{4H} J^{AM} H_{MBP} \hat{e}^P \end{pmatrix},$$

$$t = \begin{pmatrix} \frac{3}{2}w + \frac{1}{2}v & -\hat{e}^A \\ -\hat{e}^A & -J^A_M \omega^M_B - \frac{1}{4H} H_{ABP} \hat{e}^P + \frac{1}{2}(-w+v) \delta^A_B \end{pmatrix}.$$

Since $\text{Tr}(p) = 0$ and $t = t^T$ we have

$$p \in \mathfrak{sl}(2, \mathbb{R}), \quad \begin{pmatrix} q & t \\ -t^T & -q^T \end{pmatrix} \in \mathfrak{sp}(2n+2, \mathbb{R})$$

and, hence, the Levi-Civita connection one-form Ω takes values in the Lie algebra $\mathfrak{sp}(2n+2, \mathbb{R}) \cdot \mathfrak{sp}(2, \mathbb{R})$. By corollary 2.1 the Riemann holonomy group is given by $\text{Hol}(D) \subset \text{Sp}(2n+2, \mathbb{R}) \cdot \text{Sp}(2, \mathbb{R})$ and the manifold is para-quaternion Kähler. \square

Remark 6.1. On the para-quaternion Kähler manifold there exists the almost complex structure $\tilde{\mathbb{K}}$. This in fact coincides with the almost complex structure \mathbb{I}' defined in the previous section $\tilde{\mathbb{K}} = \mathbb{I}'$. We may also define the complex structure \mathbb{I} as in the previous section. Since the differentials of the complex vielbeins are the same for both manifolds (since they are built out of the same real vielbeins), by theorem 6.2 these complex structures are also integrable on the para-quaternion Kähler manifold. Both structures are compatible with the metric, but this time it is \mathbb{I}' that is compatible with the para-quaternion Kähler structure.

Chapter 7

Four-dimensional black holes

We will now apply the results of the previous chapter to the search for new stationary black hole solutions. The definition of stationary backgrounds is that they admit a timelike isometry of the spacetime metric. We can therefore dimensionally reduce over this redundant dimension, which corresponds to making the choice $\epsilon = 1$ in section 6.2. Our starting point is the Lagrangian (6.37) of the dimensionally-reduced theory¹

$$\begin{aligned} \mathbf{e}_3^{-1} \mathcal{L}_3 = & \frac{1}{2} R_3 - \tilde{H}_{ab} \left(\partial_\mu q^a \partial^\mu q^b - \partial_\mu \hat{q}^a \partial^\mu \hat{q}^b \right) \\ & - \frac{1}{H^2} \left(q^a \Omega_{ab} \partial_\mu q^b \right)^2 + \frac{2}{H^2} \left(q^a \Omega_{ab} \partial_\mu \hat{q}^b \right)^2 \\ & - \frac{1}{4H^2} \left(\partial_\mu \tilde{\phi} + 2\hat{q}^a \Omega_{ab} \partial_\mu \hat{q}^b \right)^2. \end{aligned} \quad (7.1)$$

This Lagrangian, despite being three-dimensional, encodes all the information about stationary solutions to the full four-dimensional theory, i.e. there is a one-to-one correspondence between instanton solutions of the above Euclidean Lagrangian and stationary solutions of the four-dimensional Lorentzian theory. Not all of these solutions describe black holes with finite scalar fields, however, so we must be careful to identify those solutions that have an event horizon, are asymptotically Minkowski and have scalar fields that take finite values.

Unlike our previous discussion about five-dimensional black holes, we will not immediately assume that solutions are static. This allows us to consider rotating solutions. We find that if the three-dimensional metric is flat then we can find solutions to generic models in terms of harmonic functions. Unfortunately these solutions do not have an event horizon, and therefore do not correspond to black holes. Using this method we recover the known rotating BPS solutions found in [62, 63, 41, 42]. We also observe that by introducing a field rotation matrix the three-dimensional metric is no longer flat, and so we cannot immediately adapt our ansatz to find non-BPS solutions. We leave a detailed study of non-BPS rotating solutions for future work.

We then investigate static solutions. We begin by looking for extremal solutions, which, from our experience of the extremal Reissner–Nordström solution (3.14), we expect will have a flat three-dimensional metric. Solutions are characterised by harmonic functions and BPS solutions can be found for generic models. We then show that, as in the five-dimensional case, in the presence of a non-trivial field rotation matrix BPS solutions can always be transformed into non-BPS solutions. However, such field rotation matrices are not guaranteed to exist for generic models. We prove that

¹We are using the compact notation $(q^a \Omega_{ab} \partial_\mu q^b)^2 = \sum_\mu (q^a \Omega_{ab} \partial_\mu q^b) (q^c \Omega_{cd} \partial^\mu q^d)$.

a non-trivial field rotation matrix always exists for axion-free configurations of a large class of models that includes all models in the image of the spacelike r-map. We can therefore construct new non-BPS solutions for this entire class of models.

Finally, we will discuss new non-extremal black hole solutions. This builds on our understanding of non-extremal black holes in the five-dimensional case, and is new and as yet unpublished work [61]. We will only consider axion-free solutions of a certain class of models previously mentioned, which is a generalisation of the class of models in the image of the r-map. We will first consider non-extremal solutions for *STU*-like models, where the second order equations of motion can be integrated explicitly. We demonstrate that for these solutions to lift to non-extremal black holes with finite scalar fields the number of integration constants must reduce by half, which is suggestive of a first order rewriting. For the *STU* model we recover the results of [104], while for all other models in the *STU*-like class these non-extremal solutions are new. We then discuss generic solutions in our class of models, and find that the number of independent scalar fields depends on the number of blocks in the metric. Since all metrics of this form have at least two blocks we find a universal solution with one non-constant physical scalar field.

In section 7.1 we calculate the field equations for the Lagrangian (7.1), and introduce a natural set of dual scalar fields. We investigate stationary solutions in section 7.2. We then move on to static solutions in section 7.3, where we present new extremal black hole solutions. We end in section 7.4 by constructing new non-extremal solutions.

7.1 Equations of motion and dual coordinates

We will now present the full set of field equations for the Lagrangian (7.1). We first perform the variation with respect to the q^a fields, and obtain the equation of motion

$$\begin{aligned}
& 2\nabla^\mu \left[\tilde{H}_{ab} \partial_\mu q^b \right] - \partial_a \tilde{H}_{bc} \left(\partial_\mu q^b \partial^\mu q^c - \partial_\mu \hat{q}^b \partial^\mu \hat{q}^c \right) \\
& + 2\nabla^\mu \left[\frac{1}{H^2} q^c \Omega_{ca} \left(q^d \Omega_{de} \partial_\mu q^e \right) \right] \\
& - 2\partial_a \left(\frac{1}{H} q^c \right) \left[\Omega_{cb} \partial_\mu q^b \frac{1}{H} \left(q^d \Omega_{de} \partial_\mu q^e \right) - 2\Omega_{cb} \partial_\mu \hat{q}^b \frac{1}{H} \left(q^d \Omega_{de} \partial_\mu \hat{q}^e \right) \right] \\
& - \partial_a \left(\frac{1}{4H^2} \right) \left(\partial_\mu \tilde{\phi} + 2\hat{q}^c \Omega_{cd} \partial_\mu \hat{q}^d \right)^2 = 0.
\end{aligned} \tag{7.2}$$

Next, by varying the \hat{q}^a fields we obtain

$$\begin{aligned}
& -2\nabla^\mu \left[\tilde{H}_{ab} \partial_\mu \hat{q}^b \right] \\
& - 4\nabla^\mu \left[\frac{1}{H^2} q^c \Omega_{ca} \left(q^d \Omega_{de} \partial_\mu \hat{q}^e \right) \right] + \nabla^\mu \left[\frac{1}{H^2} \hat{q}^b \Omega_{ba} \left(\partial_\mu \tilde{\phi} + 2\hat{q}^c \Omega_{cd} \partial_\mu \hat{q}^d \right) \right] \\
& - \frac{1}{H^2} \Omega_{ab} \partial_\mu \hat{q}^b \left(\partial_\mu \tilde{\phi} + 2\hat{q}^c \Omega_{cd} \partial_\mu \hat{q}^d \right) = 0.
\end{aligned} \tag{7.3}$$

The variation of the $\tilde{\phi}$ field, which descends from the Kaluza–Klein vector, gives us simply

$$\nabla^\mu \left[\frac{1}{4H^2} \left(\partial_\mu \tilde{\phi} + 2\hat{q}^c \Omega_{cd} \partial_\mu \hat{q}^d \right) \right] = 0. \tag{7.4}$$

Since we have dualised the Kaluza–Klein vector, which swaps the role of the field equations and Bianchi identities, this equation gives us simply the Bianchi identity for

the field strength of the Kaluza–Klein vector $V_{\mu\nu}$. Finally, from the variation of the metric we find the Einstein equations

$$\begin{aligned} & \frac{1}{2} R_{\mu\nu} - \tilde{H}_{ab} \left(\partial_\mu q^a \partial_\nu q^b - \partial_\mu \hat{q}^a \partial_\nu \hat{q}^b \right) \\ & - \frac{1}{H^2} \left(q^a \Omega_{ab} \partial_\mu q^b \right) \left(q^c \Omega_{cd} \partial_\nu q^d \right) + \frac{2}{H^2} \left(q^a \Omega_{ab} \partial_\mu \hat{q}^b \right) \left(q^c \Omega_{cd} \partial_\nu \hat{q}^d \right) \\ & - \frac{1}{4H^2} \left(\partial_\mu \tilde{\phi} + 2\hat{q}^a \Omega_{ab} \partial_\mu q^b \right) \left(\partial_\nu \tilde{\phi} + 2\hat{q}^c \Omega_{cd} \partial_\nu \hat{q}^d \right) = 0 . \end{aligned} \quad (7.5)$$

Using the Hesse potential \tilde{H} one may define a natural set of dual coordinates q_a by taking the first derivative of the Hesse potential

$$q_a := \frac{\partial \tilde{H}}{\partial q^a} . \quad (7.6)$$

From the chain rule we have

$$\partial_\mu q_a = \partial_{a,b}^2 \tilde{H} \partial_\mu q^b = \tilde{H}_{ab} \partial_\mu q^b ,$$

and so we can think of the metric as an operator that raises and lowers indices. One must be careful, however, as an additional minus sign is acquired when lowering (or raising) an index without a derivative

$$q_a = -\tilde{H}_{ab} q^b .$$

Not only do the equations of motion simplify greatly (particularly in the static case) when we use dual coordinates, they are also critical for obtaining multi-centred solutions in terms of harmonic functions. Roughly speaking, the existence of the dual coordinates provides an off-shell realisation of an integrability condition needed for the existence of multi-centred solutions. The dual coordinates can also be related to u^I, v_I (the imaginary parts of X^I, F_I) through the expression

$$q_a = -\frac{H_a}{2H} = \frac{1}{H} \begin{pmatrix} -v_I \\ u^I \end{pmatrix} . \quad (7.7)$$

We can also interpret the dual coordinates as the components of $-\xi^b$ in a special real coordinate system

$$q_a = -\xi^b \left(\frac{\partial}{\partial q^a} \right) .$$

The complex scalar fields Y^I and their conjugates F_I can now be written simply as

$$Y^I = - \left(\tilde{H}^{Ib} + i\Omega^{Ib} H \right) q_b , \quad (7.8a)$$

$$F_I = \Omega_{Ib} q^b + iH q_I . \quad (7.8b)$$

At this stage one may rewrite the equations of motion in terms of dual coordinates. However, we will wait until we impose certain ansätze on the metric and scalar fields such that many of the terms in the field equations directly cancel with one another. These ansätze fall into three distinct categories depending on the particular type of solution under consideration:

1. Isotropic ansatz, for rotating solutions.
2. Extremal ansatz, for static extremal black hole solutions.
3. Non-extremal ansatz, for static non-extremal black hole solutions.

We will discuss the precise properties of each ansatz in the next three sections.

7.2 Stationary solutions

The first class of solutions we will investigate are stationary solutions, which are also the most complicated. We will need to solve the full set of equations of motion (7.2) – (7.5). To make this job tractable we will impose what we call the isotropic ansatz. This states that the three-dimensional metric is Ricci flat (and therefore flat), which, from the Einstein equations (7.5), is equivalent to identifying isotropic distributions of the para-quaternion Kähler target manifold.

Solutions to the isotropic ansatz are found for generic models, and are characterised by harmonic functions. We will then focus on single-centred rotating (i.e. axially symmetric) solutions, which are one of the simplest types of stationary solution. Despite the fact that solutions can be found for generic models they cannot always be written down explicitly. This is because one has to invert a system of algebraic equations, which is not always possible in closed form. We end by presenting the solution to the *STU* model, which can be found explicitly.

7.2.1 Isotropic ansatz

We shall now present our isotropic ansatz, which is characterised by the requirement that the three-dimensional metric is Ricci flat (the Ricci tensor vanishes), which in three dimensions automatically implies that the metric is flat (the Riemann tensor vanishes). From the Einstein equations (7.5) it is clear that this means we must find solutions for which the energy-momentum tensor vanishes identically. Mathematically it means that we must identify distributions $D \subset TM$ in which the metric vanishes identically, i.e.

$$g_M(X, Y) = 0 \quad \forall X, Y \in \Gamma(D) .$$

Such distributions are called isotropic, which is where our ansatz gets its name.

Let us remain focussed on the Einstein equations (7.5) with a flat metric. Motivated by our experience of constructing five-dimensional black holes, we impose the first part of our isotropic ansatz, which is to make the identification

$$\partial_\mu q^a = \pm \partial_\mu \hat{q}^a , \quad (7.9)$$

where the choice of sign is fixed for all values of a . In this case the first line of the Einstein equations vanishes identically. The second and third lines reduce to

$$\frac{1}{4H^2} \left(\partial_\mu \tilde{\phi} + 2\hat{q}^a \Omega_{ab} \partial_\mu q^b \right)^2 = \frac{1}{H^2} \left(q^a \Omega_{ab} \partial_\mu q^b \right)^2 .$$

The second part of our isotropic ansatz is now clear: we make the identification

$$\frac{1}{2} \left(\partial_\mu \tilde{\phi} + 2\hat{q}^a \Omega_{ab} \partial_\mu \hat{q}^b \right) = q^a \Omega_{ab} \partial_\mu q^b , \quad (7.10)$$

where the choice of sign is important. Solving for $\tilde{\phi}$ we have

$$\tilde{\phi} = 2(q^a \mp \hat{q}^a) \Omega_{ab} q^b ,$$

where $q^a \mp \hat{q}^a$ is constant in spacetime due to the first part of the ansatz (7.9).

Proposition 7.1 . After imposing the isotropic ansatz (7.9) and (7.10) all equations of motion reduce to the Laplace equation

$$\Delta q_a = 0 , \quad (7.11)$$

where q_a are the dual coordinates introduced in the previous section.

Proof. By definition of the isotropic ansatz the Einstein equations are solved with a flat spacetime metric. Let us now consider the equation of motion (7.4), which corresponds to the variation $\tilde{\phi}$. After imposing (7.9) and (7.10) this becomes simply

$$\nabla^\mu \left[\frac{1}{H^2} \left(q^a \Omega_{ab} \partial_\mu q^b \right) \right] = 0 . \quad (7.12)$$

We will come back to this expression shortly. Next, consider the equation of motion (7.2), which corresponds to the variation of q^a . When we impose (7.9) and (7.10) the second term vanishes, the third line simplifies, and due to (7.12) the derivative in the second line will only act on q^c . We are left with

$$\begin{aligned} 2\nabla^\mu \left[\tilde{H}_{ab} \partial_\mu q^b \right] + \frac{2}{H^2} \partial_\mu q^c \Omega_{ca} \left(q^d \Omega_{de} \partial^\mu q^e \right) + 2\partial_a \left(\frac{1}{H} q^c \right) \Omega_{cb} \partial_\mu q^b \frac{1}{H} \left(q^d \Omega_{de} \partial^\mu q^e \right) \\ - 2\partial_a \left(\frac{1}{H} \right) q^c \Omega_{cb} \partial_\mu q^b \frac{1}{H} \left(q^d \Omega_{de} \partial^\mu q^e \right) = 0 . \end{aligned}$$

It is clear that the fourth term cancels with the derivative acting on the Hesse potential in the third term, and so we have

$$2\nabla^\mu \left[\tilde{H}_{ab} \partial_\mu q^b \right] + \left(\frac{2}{H^2} \partial_\mu q^c \Omega_{ca} + 2\frac{1}{H^2} \Omega_{ab} \partial_\mu q^b \right) \left(q^d \Omega_{de} \partial^\mu q^e \right) = 0 .$$

Due to the antisymmetry of Ω_{ab} the second and third terms cancel, and writing the first term in terms of the dual coordinates q_a we are left with the Laplace equation (7.11). We now consider the equation of motion (7.3), which corresponds to the variation of \hat{q}^a . Using (7.12) the derivative in the second term will only act on q^c , and the second and third terms simplify

$$-2\nabla^\mu \left[\tilde{H}_{ab} \partial_\mu \hat{q}^b \right] - \left(\frac{2}{H^2} \partial_\mu q^c \Omega_{ca} + \frac{2}{H^2} \Omega_{ab} \partial_\mu q^b \right) \left(q^d \Omega_{de} \partial_\mu q^e \right) = 0 .$$

From the antisymmetry of Ω_{ab} the second and third terms cancel, leaving us again with the Laplace equation (7.11). Finally, we revisit the equation of motion for $\tilde{\phi}$, which we have already reduced to (7.12). Using the identity $q^a \Omega_{ac} = -\frac{1}{4} H_a \Omega^{ab} H_{bc}$ we can write the left hand side as

$$\begin{aligned} \nabla^\mu \left[\frac{1}{H^2} \left(q^a \Omega_{ab} \partial_\mu q^b \right) \right] &= -\frac{1}{4} \nabla^\mu \left[\frac{1}{H^2} \left(H_a \Omega^{ab} H_{bc} \partial_\mu q^c \right) \right] \\ &= -\nabla^\mu \left[\tilde{H}_a \Omega^{ab} \tilde{H}_{bc} \partial_\mu q^c \right] \\ &= -q_a \Omega^{ab} \Delta q_b . \end{aligned}$$

This clearly vanishes for solutions of the Laplace equation (7.11), which completes the proof. \square

We have shown that the isotropic ansatz is solved by (7.9), (7.10) and (7.11). Solutions are characterised by the harmonic functions

$$\mathcal{H}_a := \begin{pmatrix} -\mathcal{H}_I \\ \mathcal{H}^I \end{pmatrix} , \quad (7.13)$$

where the scalar fields are given in terms of these harmonic functions by

$$q_a = \pm \mathcal{H}_a , \quad (7.14)$$

$$\partial_\mu \hat{q}^a = \tilde{H}^{ab} \partial_\mu \mathcal{H}_b . \quad (7.15)$$

It is worth noting that these solutions may not be the most general isotropic solutions. As we have seen, for these particular solutions the first line of the Einstein equations vanishes independently of the second and third lines, and one could conceive of a solution in which these terms do not vanish independently. We leave the investigation of more complicated isotropic solutions for future work.

It is instructive to consider our isotropic ansatz in terms of four-dimensional fields. The q^a coordinates can be written as

$$q^a = \frac{1}{2} e^{\phi/2} \begin{pmatrix} X^I + \bar{X}^I \\ F_I + \bar{F}_I \end{pmatrix} ,$$

and the derivatives of the \hat{q}^a coordinates can be written as

$$\partial_\mu \hat{q}^a = \frac{1}{2} \begin{pmatrix} F_{\mu 0}^I \\ G_{I|\mu 0} \end{pmatrix} .$$

We can then write the first part of the isotropic ansatz (7.9) in terms of four-dimensional fields as

$$\partial_\mu (e^{\phi/2} (X^I + \bar{X}^I)) = \pm F_{\mu 0}^I = \pm (F_{\mu 0}^{I+} + F_{\mu 0}^{I-}) , \quad (7.16a)$$

$$\partial_\mu (e^{\phi/2} (F_I + \bar{F}_I)) = \pm G_{I|\mu 0} = \pm (G_{I|\mu 0}^+ + G_{I|\mu 0}^-) . \quad (7.16b)$$

For supersymmetric solutions these are the well known BPS equations that follow from the requirement that the variation of the gaugino vanishes. See for example [42]. In our formalism they correspond to Bogomol'nyi equations associated with the first line of the three-dimensional Lagrangian (7.1). Let us now consider the Laplace equation for the dual coordinates (7.11). Using the expression (7.7) we can write the dual coordinates in terms of four-dimensional fields as

$$q_a = i e^{-\phi/2} \begin{pmatrix} -(F_I - \bar{F}_I) \\ X^I - \bar{X}^I \end{pmatrix} ,$$

and so the solution reads

$$e^{-\phi/2} (X^I - \bar{X}^I) = -i \mathcal{H}^I , \quad (7.17a)$$

$$e^{-\phi/2} (F_I - \bar{F}_I) = -i \mathcal{H}_I . \quad (7.17b)$$

These are nothing other than the well known black hole stabilisation equations. Here they were derived directly from the field equations by imposing the isotropic ansatz (7.9) and (7.10). They reduce to the attractor equations in the near horizon limit.

Let us end this section by commenting on U(1) gauge fixing. The Lagrangian and equations of motion are invariant under the U(1) isometry, which up until now we had not fixed. The isotropic ansatz (7.9) and (7.10) clearly breaks this symmetry, as it relates the fields q^a , which transform under the isometry, with the fields $\hat{q}^a, \tilde{\phi}$, which do not. Therefore the isotropic ansatz implicitly fixes the U(1) symmetry, and it does so in a symplectically covariant way. The key was that we had to impose part of the field equations. We can actually use the solution (7.11) to write the gauge fixing condition explicitly. From the equations (7.17a) and (7.17b) and the D-gauge (6.25) we can write

$$X^I \mathcal{H}_I - F_I \mathcal{H}^I = e^{-\phi/2} . \quad (7.18)$$

This equation characterises the U(1) gauge fixing condition.

7.2.2 Rotating solutions

The isotropic ansatz, presented above, provides stationary solutions for generic models with flat three-dimensional metrics. These solutions are characterised by harmonic functions (7.13) through (7.14) and (7.15). We will now focus on the subset of solutions that are:

1. Single-centred,
2. Axially symmetric.

Such solutions represent rotating spacetimes. Unfortunately we will find that they always contain a naked singularity, and can therefore not be called black holes. In order to find legitimate rotating black hole solutions we must relax the condition that the three-dimensional metric is flat. As this is a significant increase in the complexity of our analysis we leave it for future work.

The solutions to the isotropic ansatz are given in terms of the dual coordinates q_a , and it is not always possible to write the solutions explicitly for the original four-dimensional fields. To do so involves inverting the algebraic relations (7.6), and corresponds to solving the generalised stabilisation equations. This is always possible by the inverse function theorem because \tilde{H}_{ab} is invertible, but in practice it is not always possible to solve such a set of equations in a closed form. In fact it is only possible for a few simple examples. For this reason we will focus on the *STU* model, where explicit solutions can be found.

The first task is to disentangle the four-dimensional metric from the solution given by (7.10), (7.13), (7.14) and (7.15). To do so one simply retraces the dimensional reduction procedure to find

$$g_{\mu\nu} = \delta_{\mu\nu} , \quad (7.19a)$$

$$e^\phi = -2H , \quad (7.19b)$$

$$\partial_\mu V_\nu = \frac{1}{2} \varepsilon_{\mu\nu\rho} (\mathcal{H}_I \partial^\rho \mathcal{H}^I - \mathcal{H}^I \partial^\rho \mathcal{H}_I) . \quad (7.19c)$$

The first equation simply tells us that the three-dimensional metric is flat. The second equation is model dependent, and so we will postpone the discussion of this until later. Let us focus on the third equation, (7.19c), which is independent of the particular model. Since we are now focussing on axially symmetric solutions it is natural to use a oblate spheroidal coordinates (a.k.a. Boyer–Lindquist coordinates), as discussed in section 3.2.2, which are related to Cartesian coordinates through

$$\begin{aligned} x &= \sqrt{r^2 + \alpha^2} \sin \theta \cos \varphi , \\ y &= \sqrt{r^2 + \alpha^2} \sin \theta \sin \varphi , \\ z &= r \cos \theta . \end{aligned}$$

At this stage α is simply a constant parameter. Following the method for producing rotating solutions outlined in [63, 41] we impose that solutions are axially symmetric about the z coordinate. In these coordinates the flat three-dimensional metric is written as

$$ds_3^2 = \left(\frac{r^2 + \alpha^2 \cos^2 \theta}{r^2 + \alpha^2} \right) dr^2 + (r^2 + \alpha^2 \cos^2 \theta) d\theta^2 + (r^2 + \alpha^2) \sin^2 \theta d\varphi^2 .$$

We can write (7.19c) in oblate spheroidal coordinates as

$$\frac{1}{(r^2 + \alpha^2) \sin \theta} \partial_\theta V_\varphi = \frac{1}{2} (\mathcal{H}_I \partial_r \mathcal{H}^I - \mathcal{H}^I \partial_r \mathcal{H}_I) , \quad (7.20)$$

$$-\frac{1}{\sin \theta} \partial_r V_\varphi = \frac{1}{2} (\mathcal{H}_I \partial_\theta \mathcal{H}^I - \mathcal{H}^I \partial_\theta \mathcal{H}_I) . \quad (7.21)$$

General single-centred harmonic functions in oblate spheroidal coordinates take the form

$$\begin{aligned} \mathcal{H}^I &= h^I + \frac{p^I r + m^I \alpha \cos \theta}{R} , \\ \mathcal{H}_I &= h_I + \frac{q_I r + m_I \alpha \cos \theta}{R} , \end{aligned}$$

where $R = r^2 + \alpha^2 \cos^2 \theta$, and $(h^I, h_I, m^I, m_I, p^I, q_I)$ are independent integration constants. We can interpret h^I, h_I as the values of the scalars at infinity, p^I, q_I as the magnetic and electric charges, and m^I, m_I as the dipole momenta [41].

We can integrate equations (7.20) and (7.21) to find an explicit expression for the only non-zero component of the KK-vector

$$\begin{aligned} V_\varphi &= \frac{1}{2} (h_I p^I - h^I q_I) \cos \theta \left(\frac{r^2 + \alpha^2}{R} \right) \\ &\quad + \frac{\alpha}{2} (m_I h^I - m^I h_I) \sin^2 \theta \left(\frac{r}{R} \right) \\ &\quad + \frac{\alpha}{4} (m_I p^I - m^I q_I) \sin^2 \theta \left(\frac{1}{R} \right) + C , \end{aligned} \quad (7.22)$$

where C is an arbitrary constant. By gathering together the integration constants to form the three symplectic vectors $(h^I, h_I)^T$, $(p^I, q_I)^T$ and $(m^I, m_I)^T$, one observes that all three symplectic products appear in the above expression. One can interpret the coefficient of the first line as the NUT charge $n = \frac{1}{2} (h_I p^I - h^I q_I)$ by comparison with [109, 110], and the coefficient of the second line as the angular momentum $l = \frac{\alpha}{2} (m_I h^I - m^I h_I)$. The coefficient of the third line does not have a specific interpretation, but has previously been observed in rotating solutions [109].

In order for the solution to be asymptotically flat the field strength of the KK-vector must vanish at radial infinity. By inspection of (7.22) one can see that this is equivalent to a vanishing NUT charge

$$n = \frac{1}{2} (h_I p^I - h^I q_I) = 0 . \quad (7.23)$$

This places one constraint on the integration constants (h^I, h_I) . The only other requirement for asymptotic flatness is that the KK-scalar approaches unity at radial infinity

$$e^\phi \xrightarrow{r \rightarrow \infty} 1 . \quad (7.24)$$

We will see that this places another constraint on the integration constants (h^I, h_I) . This makes sense because the solution at radial infinity should only depend on the asymptotic values of the physical scalar fields z^A , which have two fewer real dimensions than (h^I, h_I) .

We will now show that the solutions described by the isotropic ansatz are always BPS. We do so by explicitly computing the ADM mass and showing that it equals the

central charge. The formula for the ADM mass for axially symmetric solutions is given by

$$\begin{aligned} 16\pi M_{\text{ADM}} &= 2 \oint_{S_\infty^2} d^2\Sigma^r e^{-\phi} \partial_r \phi \\ &= -2 \lim_{r \rightarrow \infty} \oint_{S_r^2} d^2\Sigma^r \partial_r e^{-\phi} . \end{aligned}$$

The measure is given by

$$\begin{aligned} d^2\Sigma^r &= \varepsilon_{r\theta\varphi} d\theta \wedge d\varphi \\ &= (r^2 + \mathcal{O}(r)) \sin \theta \, d\theta \wedge d\varphi , \end{aligned}$$

and we can expand the KK-scalar as

$$e^{-\phi} = 1 + \mathcal{O}\left(\frac{1}{r}\right) .$$

Plugging this into the formula for the ADM mass we have

$$\begin{aligned} 16\pi M_{\text{ADM}} &= -2 \lim_{r \rightarrow \infty} \oint_{S_r^2} (r^2 + \mathcal{O}(r)) \sin \theta \, d\theta d\varphi \, \partial_r e^{-\phi} \\ &= -2 \lim_{r \rightarrow \infty} \oint_{S_1^2} \sin \theta \, d\theta d\varphi \, r^2 \partial_r e^{-\phi} . \end{aligned}$$

We therefore find that the ADM mass has a particularly simple dependence on the Hesse potential

$$\begin{aligned} M_{\text{ADM}} &= - \lim_{r \rightarrow \infty} r^2 \partial_r \tilde{H} \\ &= \lim_{r \rightarrow \infty} r^2 q^a \partial_r q_a . \end{aligned}$$

Since the NUT charge vanishes one has $r^2 q^a \Omega_{ab} \partial_r q^b \rightarrow 0$ asymptotically, which implies that $r^2 q_a \Omega^{ab} \partial_r q_b \rightarrow 0$. We can then write the mass as

$$\begin{aligned} M_{\text{ADM}} &= \lim_{r \rightarrow \infty} r^2 \left(q^a - iH \Omega^{ab} q_b \right) \partial_r q_a \\ &= \lim_{r \rightarrow \infty} |X^I q_I - F_I p^I| = \lim_{r \rightarrow \infty} |\mathcal{Z}| , \end{aligned} \tag{7.25}$$

which confirms that these solutions are BPS.

This is all we can say for generic models. To find explicit solutions for all the fields it remains to write equation (7.19b) explicitly, which involves finding an expression for the Hesse potential in terms of the dual coordinates, and to invert the relation (7.6). This can be done explicitly for the *STU* model.

7.2.3 The *STU* model

As briefly mentioned in section 3.2.3, the *STU* model is characterised by the prepotential

$$F = - \frac{Y^1 Y^2 Y^3}{Y^0} .$$

The Hesse potential for the STU model in terms of u^I, v_I is given by (see appendix B)

$$H = -2 \left(- (u.v)^2 + 4v_1 u^1 v_2 u^2 + 4v_1 u^1 v_3 u^3 + 4v_2 u^2 v_3 u^3 + 4u^0 v_1 v_2 v_3 - 4v_0 u^1 u^2 u^3 \right)^{1/2}. \quad (7.26)$$

The isotropic solution for the q^a fields is given in terms of harmonic functions by (7.13) and (7.14), and can be formulated in terms of u^I, v_I and the Hesse potential as²

$$\frac{1}{H} u^I = \mathcal{H}^I, \quad \frac{1}{H} v_I = \mathcal{H}_I.$$

From the expression $e^\phi = -2H$ we can write the KK-scalar for the STU model explicitly in terms of harmonic functions

$$e^{-\phi} = \left(- (\mathcal{H}^I \mathcal{H}_I)^2 + 4\mathcal{H}_1 \mathcal{H}^1 \mathcal{H}_2 \mathcal{H}^2 + 4\mathcal{H}_1 \mathcal{H}^1 \mathcal{H}_3 \mathcal{H}^3 + 4\mathcal{H}_2 \mathcal{H}^2 \mathcal{H}_3 \mathcal{H}^3 + 4\mathcal{H}^0 \mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_3 - 4\mathcal{H}_0 \mathcal{H}^1 \mathcal{H}^2 \mathcal{H}^3 \right)^{1/2}. \quad (7.27)$$

Using this formula we can find explicit expressions for u^I, v_I in terms of harmonic functions

$$u^I = -\frac{1}{2} e^{\phi/2} \mathcal{H}^I, \quad v_I = -\frac{1}{2} e^{\phi/2} \mathcal{H}_I. \quad (7.28)$$

To ensure that the solution is asymptotically Minkowski we must impose the constraints (7.23) and (7.24), which for this model correspond to

$$h_I p^I - h^I q_I = 0, \quad (7.29a)$$

$$-(h^I h_I)^2 + 4h_1 h^1 h_2 h^2 + 4h_1 h^1 h_3 h^3 + 4h_2 h^2 h_3 h^3 + 4h^0 h_1 h_2 h_3 - 4h_0 h^1 h^2 h^3 = 1. \quad (7.29b)$$

The KK-vector for the solution is given by (7.22), and together with (7.27) and (7.19a) the four-dimensional line-element can be constructed according to the decomposition (6.28).

We must also write the original four-dimensional complex scalar fields $X^I = e^{-\frac{\phi}{2}} Y^I$ in terms of the harmonic functions. To do this we could use the expression (7.8a), which involves calculating the inverse metric and is rather involved. Luckily we can adapt the results of [111] to obtain the result directly³

$$\begin{aligned} Y^0 &= \frac{1}{U + \bar{U}} (2u^3 + i2u^0 \bar{U}), & Y^1 &= \frac{1}{U + \bar{U}} (-2v_2 + i2u^1 \bar{U}), \\ Y^2 &= \frac{1}{U + \bar{U}} (-2v_1 + i2u^2 \bar{U}), & Y^3 &= iU Y^0, \end{aligned} \quad (7.30)$$

where

$$U := i \frac{v_0 u^0 + v_1 u^1 + v_2 u^2 - v_3 u^3}{2(v_3 u^0 + u^1 v^1)} \pm \sqrt{\frac{v_1 v_2 - v_0 u^3}{v_3 u^0 + u^1 u^2} - \frac{(v_0 u^0 + v_1 u^1 + v_2 u^2 - v_3 u^3)^2}{4(v_3 u^0 + u^1 u^2)^2}}. \quad (7.31)$$

Finally, we may obtain expressions for the gauge field strengths using (7.15) and (6.35), or alternatively using (7.16).

²We will take the positive sign in equations (7.14), but it is understood that either sign is equivalent.

³In this reference they consider the near horizon limit in which case u^I and v_I are given by the constant magnetic and electric charges p^I and q_I , but it is the algebraic expressions themselves that we are interested in.

7.3 Static extremal black holes

7.3.1 Extremal ansatz

We will now present an ansatz for finding extremal solutions. It is essentially the same as the isotropic ansatz presented in the previous section but restricted to static four-dimensional spacetimes. There is a subtle difference, however, in that the extremal ansatz allows a generalisation which produces non-BPS extremal solutions.

As for the isotropic ansatz we try to identify field configurations in which the three-dimensional metric is flat, and, hence, we are again looking to identify isotropic submanifolds of the para-quaternion Kähler target manifold. This time we impose that the four-dimensional metric is static, which in three-dimensional fields means that the KK-vector vanishes. This corresponds to setting

$$\frac{1}{2H} \left(\partial_\mu \tilde{\phi} + 2\hat{q}^a \Omega_{ab} \partial_\mu \hat{q}^b \right) = 0 . \quad (7.32)$$

We now impose the first part of our extremal ansatz

$$q^a \Omega_{ab} \partial_\mu q^b = q^a \Omega_{ab} \partial_\mu \hat{q}^b = 0 , \quad (7.33)$$

which, given (7.32), is actually equivalent to imposing the second part of the isotropic ansatz from the previous section in static backgrounds. Let us investigate the effect this has on the equations of motion. All terms containing a factor of the above expressions will vanish, and only the first line of (7.2), (7.3) and (7.5) is relevant (equation (7.4) is clearly solved automatically). We are left with the truncated equations of motion

$$\nabla^\mu \left[\tilde{H}_{ab} \partial_\mu q^b \right] - \frac{1}{2} \partial_a \tilde{H}_{bc} \left(\partial_\mu q^b \partial^\mu q^c - \partial_\mu \hat{q}^b \partial^\mu \hat{q}^c \right) = 0 , \quad (7.34)$$

$$\nabla^\mu \left[\tilde{H}_{ab} \partial_\mu \hat{q}^b \right] = 0 , \quad (7.35)$$

$$\tilde{H}_{ab} \left(\partial_\mu q^a \partial_\nu q^b - \partial_\mu \hat{q}^a \partial_\nu \hat{q}^b \right) = \frac{1}{2} R_{\mu\nu} . \quad (7.36)$$

It can be easily shown that these equations of motion derive from an effective action, which is given by the first line of (7.1)

$$\mathbf{e}_3^{-1} \mathcal{L}_3 = \frac{1}{2} R - \tilde{H}_{ab} \left(\partial_\mu q^a \partial^\mu q^b - \partial_\mu \hat{q}^a \partial^\mu \hat{q}^b \right) .$$

Recall that the Lagrangian (7.1) and equations of motion (7.2) – (7.5) are still invariant under U(1) phase transformations. The static ansatz (7.33) implicitly breaks this symmetry as it relates the q^a coordinates, which transform under this symmetry, with the \hat{q}^a coordinates, which do not. This gauge fixing condition is characterised by the expression

$$x^I G_{I|\mu\nu} - y_I F_{\mu\nu}^I = 0 . \quad (7.37)$$

Let us focus on the Einstein equation (7.36) with a flat three-dimensional metric. In order to solve this we impose the second part of our extremal ansatz

$$\partial_\mu q^a = \pm \partial_\mu \hat{q}^a , \quad (7.38)$$

where the choice of sign is fixed for all a . This ansatz will lead to BPS extremal black hole solutions. We will later see that this part of the ansatz can be generalised to include non-BPS solutions for models that allow for a field rotation matrix, but for

now we focus simply on BPS solutions. After imposing (7.38) one can see by inspection that the equations of motion (7.34), (7.35) and (7.36) reduce to the Laplace equation when written in terms of dual coordinates

$$\Delta q_a = 0 . \quad (7.39)$$

Solutions are characterised in terms of harmonic functions

$$\mathcal{H}_a := \begin{pmatrix} -\mathcal{H}_I \\ \mathcal{H}^I \end{pmatrix} , \quad (7.40)$$

by setting

$$q_a = \pm \mathcal{H}_a , \quad (7.41)$$

$$\partial_\mu \hat{q}^a = \tilde{H}^{ab} \partial_\mu \mathcal{H}_b . \quad (7.42)$$

For the extremal case we will consider multicentred harmonic functions of the form

$$\mathcal{H}^I = h^I + \sum_\alpha \frac{p_\alpha^I}{|x - x_\alpha|} , \quad (7.43a)$$

$$\mathcal{H}_I = h_I + \sum_\alpha \frac{q_{I\alpha}}{|x - x_\alpha|} . \quad (7.43b)$$

The first part of the ansatz (7.33) places one constraint on the integration constants

$$h_I p^I - h^I q_I = 0 . \quad (7.44)$$

We also need the corresponding four-dimensional solution to be asymptotically Minkowski, i.e. at radial infinity

$$e^\phi \xrightarrow[r \rightarrow \infty]{} 1 , \quad (7.45)$$

which places another constraint on the h^I, h_I integration constants. We therefore have a total of $2n$ integration constants from $2n$ second order equations of motion of the physical scalar fields. This suggests that solutions always satisfy first order equations. This is clear from the fact that the solutions solve (7.38).

We will now compute the ADM mass of the extremal solution (7.41), (7.42), and show that it is equal to the central charge. This indicates that the solutions we have found are always BPS solutions. The formula for the ADM mass for is given by

$$\begin{aligned} 16\pi M_{\text{ADM}} &= 2 \oint_{S_\infty^2} d^2 \Sigma^\mu e^{-\phi} \partial_\mu \phi \\ &= -2 \oint_{S_\infty^2} d^2 \Sigma^\mu \partial_\mu e^{-\phi} . \end{aligned}$$

Here we used the fact that $e^\phi \rightarrow 1$ at spatial infinity. We can write this in terms of the Hesse potential as

$$\begin{aligned} M_{\text{ADM}} &= -\frac{1}{4\pi} \oint_{S_\infty^2} d^2 \Sigma^\mu \partial_\mu \tilde{H} \\ &= \frac{1}{4\pi} \oint_{S_\infty^2} d^2 \Sigma^\mu q^a \partial_\mu q_a . \end{aligned}$$

From (7.33) we have that $q^a \Omega_{ab} \partial_\mu q^b = 0$, which we can write in terms of dual coordinates as $q_a \Omega^{ab} \partial_\mu q_b = 0$. We can then write the ADM mass as

$$\begin{aligned} M_{\text{ADM}} &= \frac{1}{4\pi} \oint_{S_\infty^2} d^2 \Sigma^\mu \left(q^a - i H \Omega^{ab} q_b \right) \partial_\mu q_a \\ &= \frac{1}{4\pi} \oint_{S_\infty^2} d^2 \Sigma^\mu |X^I \partial_\mu \mathcal{H}_I - F_I \partial_\mu \mathcal{H}^I| = |\mathcal{Z}_\infty|. \end{aligned} \quad (7.46)$$

These solutions are therefore always BPS.

We will now investigate various models for which multi-centred extremal black hole solutions can be obtained explicitly using the procedure outlined above.

7.3.2 The *STU* model

The first model we will consider is the *STU* model. The solution (7.41) relates the harmonic functions (7.43) to the dual coordinates u^I, v_I through the expressions

$$\frac{1}{H} u^I = \mathcal{H}^I, \quad \frac{1}{H} v_I = \mathcal{H}_I.$$

As for the rotating case, we can plug these equations into the formula for the Hesse potential (7.26) and use the relation $e^\phi = -2H$ to write the KK-scalar explicitly in terms of harmonic functions (7.27). This in turn gives us the explicit expressions for u^I, v_I in terms of the multi-centred harmonic functions

$$u^I = -\frac{1}{2} e^{\phi/2} \mathcal{H}^I, \quad v_I = -\frac{1}{2} e^{\phi/2} \mathcal{H}_I. \quad (7.47)$$

The requirement that solutions are static ensures that the KK-vector vanishes, while the extremal ansatz by construction ensures that the three-dimensional metric is flat

$$g_{\mu\nu} = \delta_{\mu\nu}, \quad V_\mu = 0.$$

These expressions together with (7.27) completely determine the four-dimensional metric (6.28). As for the rotating case the asymptotic integration constants h^I, h_I satisfy the two constraints (7.44) and (7.45)

$$\begin{aligned} h_I p^I - h^I q_I &= 0, \\ -(h^I h_I)^2 + 4h_1 h^1 h_2 h^2 + 4h_1 h^1 h_3 h^3 + 4h_2 h^2 h_3 h^3 \\ &\quad + 4h^0 h_1 h_2 h_3 - 4h_0 h^1 h^2 h^3 = 1. \end{aligned}$$

The original four-dimensional scalar fields are given by

$$X^I = e^{-\frac{\phi}{2}} Y^I,$$

where Y^I are given in terms of harmonic functions through (7.30) and (7.47). The gauge fields can be determined through the expressions (7.42) or alternatively (7.16).

Note that one may use the rotating solution of the *STU* model given in the previous section to obtain a static extremal black hole in the static limit $\alpha \rightarrow 0$. However, in this case one ends up with the single-centred solution

$$\mathcal{H}^I = h^I + \frac{p^I}{r}, \quad \mathcal{H}_I = h_I + \frac{q_I}{r}.$$

7.3.3 Models of the form $F = \frac{f(Y^1, \dots, Y^n)}{Y^0}$

The second class of models we shall consider are models with prepotential of the form

$$F(Y) = \frac{f(Y^1, \dots, Y^n)}{Y^0}, \quad (7.48)$$

where f is the analytic extension of a real function, or in other words f is real when evaluated on real fields. Since the prepotential F is homogeneous of degree two it follows that f is homogeneous of degree three. Note that this includes the class of prepotentials in the image of the spacelike supergravity r-map, which are given by taking f to be a homogeneous polynomial of degree three with real coefficients (see section 4.3). Moreover, if we consider the generalised spacelike r-map then the class of models described by (7.48) precisely corresponds to those obtained by reducing five-dimensional models where the Hesse potential is an arbitrary homogeneous function of degree three (not necessarily a polynomial). Such models will play an important role in our discussion of non-BPS and non-extremal solutions, which we will discuss later.

For this class of models we will only consider a restricted class of field configurations that are defined by setting Y^A purely imaginary and Y^0 purely real⁴. We will refer to such configurations as ‘axion-free’ because in this case the physical scalar fields $z^A = Y^A/Y^0$ are purely imaginary, and from models in the image of the r-map these restrictions imply that all axion-like scalars vanish (see section 4.3). The requirement can be written in terms of special real coordinates as

$$y_0 = x^A = 0, \quad (7.49)$$

which immediately implies that

$$u^0 = v_A = 0. \quad (7.50)$$

The equations of motion simplify greatly if we impose the additional condition that the corresponding parts of the gauge fields are constant

$$d\tilde{\zeta}_0 = d\zeta^A = 0, \quad (7.51)$$

which from the decomposition of the metric (7.68), which shall be discussed later, implies

$$\tilde{H}_{(n+1)b} d\hat{q}^b = \tilde{H}_{Ab} d\hat{q}^b = 0. \quad (7.52)$$

From now on whenever we refer to axion-free configurations we will assume that the conditions (7.49) and (7.51) are satisfied⁵. Geometrically speaking these conditions select totally geodesic distributions in the para-quaternion Kähler target manifold that have precisely half the number of dimensions.

The axion-free condition allows us to write Y^I, F_I explicitly in terms of u^I, v_I through the expressions

$$\begin{aligned} Y^0 &= \lambda, & F_0 &= i v_0, \\ Y^A &= i u^A, & F_A &= -\frac{f_A(u^1, \dots, u^n)}{\lambda}, \end{aligned} \quad (7.53)$$

⁴We are using the index convention that $I, J \in \{0, \dots, n\}$ whereas $A, B \in \{1, \dots, n\}$.

⁵Technically speaking while the restrictions (7.49) imply that configurations are axion-free the converse is not true. We will not consider such configurations in this thesis, however.

where $f_A = \frac{\partial f}{\partial Y^A}$ and

$$\lambda := -\sqrt{\frac{f(u^1, \dots, u^n)}{v_0}}.$$

The real parts of Y^I, F_I can be read off from (7.53) as

$$x^0 = \lambda, \quad y_A = \frac{f_A(u^1, \dots, u^n)}{\lambda}.$$

This is equivalent to solving the stabilisation equations and is the reason why we can find explicit axion-free solutions for this class of models.

At the level of the equations of motion the axion-free conditions mean that extremal solutions are characterised by only half the number of harmonic functions

$$\mathcal{H}^A = h^A + \sum_{\alpha} \frac{p_{\alpha}^A}{|x - x_{\alpha}|}, \quad \mathcal{H}_0 = h_0 + \sum_{\alpha} \frac{q_{0\alpha}}{|x - x_{\alpha}|}, \quad (7.54)$$

and are therefore charged electrically under one U(1) gauge field and magnetically under the n remaining U(1) gauge fields. In addition the first part of the extremal ansatz (7.33) is solved automatically, which suggests that the axion-free condition is not appropriate for rotating solutions. The static extremal solution for these configurations is given in terms of the harmonic functions (7.54) by (7.41) and (7.42). Using the expression for the Hesse potential (B.7) we can write the KK-scalar in terms of the harmonic functions as

$$e^{-\phi} = \sqrt{4\mathcal{H}_0 f(\mathcal{H}^1, \dots, \mathcal{H}^n)}. \quad (7.55)$$

Since the three-dimensional metric is flat and the KK-vector vanishes this is all we need to determine the four-dimensional metric through (6.28). In order for the solution to be asymptotically Minkowski the integration constants h^I, h_I must therefore satisfy the constraint

$$4h_0 f(h^1, \dots, h^n) = 1.$$

We can write u^A, v_0 explicitly in terms of harmonic functions as

$$u^A = -\frac{1}{2}e^{\phi}\mathcal{H}^A, \quad v_0 = -\frac{1}{2}e^{\phi}\mathcal{H}_0. \quad (7.56)$$

The original four-dimensional scalar fields $X^I = e^{-\frac{\phi}{2}}Y^I$ can be written in terms of harmonic functions through (7.53). The gauge fields can be determined through the expressions (7.42) or (7.16).

7.3.4 The $STU + aU^3$ model

We now turn to a specific one-parameter family of models of the form described in the last section, which are characterised by the prepotential

$$F(Y) = -\frac{Y^1 Y^2 Y^3 + a(Y^1)^3}{Y^0}.$$

This is a well-known deformation of the STU -model in which the corresponding projective special Kähler manifold is no longer a symmetric space. After imposing the

axion-free conditions (7.49) and (7.51), we may use (7.53) to write the complex scalar fields Y^I, F_I in terms of u^I, v_I as

$$\begin{aligned} Y^0 &= \lambda, & F_0 &= iv_0, \\ Y^1 &= iu^1, & F_1 &= \frac{u^2 u^3 + 3a(u^1)^2}{\lambda}, \\ Y^2 &= iu^2, & F_2 &= \frac{u^1 u^3}{\lambda}, \\ Y^3 &= iu^3, & F_3 &= \frac{u^1 u^2}{\lambda}, \end{aligned} \tag{7.57}$$

where

$$\lambda = -\sqrt{-\frac{u^1 u^2 u^3 + a(u^1)^3}{v_0}}.$$

The KK-scalar is given in terms of the harmonic functions (7.54) by

$$e^{-\phi} = \sqrt{-4\mathcal{H}_0(\mathcal{H}^1 \mathcal{H}^2 \mathcal{H}^3 + a(\mathcal{H}^1)^3)}, \tag{7.58}$$

and recall that by the extremal ansatz the three-dimensional metric is flat and the KK-vector vanishes. The asymptotic integration constants h^I, h_I must satisfy the constraint

$$-4h_0(h^1 h^2 h^3 + a(h^1)^3) = 1.$$

We can write v_0, u^A explicitly in terms of the harmonic functions using as

$$u^A = -\frac{1}{2}e^{\phi}\mathcal{H}^A, \quad v_0 = -\frac{1}{2}e^{\phi}\mathcal{H}_0, \tag{7.59}$$

and the original four-dimensional complex scalar fields $X^I = e^{-\frac{\phi}{2}}Y^I$ can be determined using (7.57). The gauge fields are given by the expressions (7.42) or (7.16).

7.3.5 *STU*-like models

We will now consider models with prepotential of the form

$$F = -\frac{(Y^1 \dots Y^n)^{\frac{3}{n}}}{Y^0}.$$

Such models are direct generalisations of the *STU* model, similar to the generalisation we considered for the five-dimensional *STU* model in chapter 5. They share the important feature that for axion-free configurations the metric is diagonal. Note that by taking $n = 3$ we recover the *STU* model. This class of model is clearly of the form described in section 7.3.3, and we can therefore find explicit axion-free solutions. These models will be particularly important later when we consider non-extremal black holes.

We restrict ourselves to axion-free configurations by imposing (7.49) and (7.51), in which case we use (7.53) to write the complex scalar fields Y^I, F_I in terms of u^I, v_I as

$$\begin{aligned} Y^0 &= \sqrt{-\frac{(u^1 \dots u^n)^{\frac{3}{n}}}{v_0}}, & F_0 &= iv_0, \\ Y^A &= iu^A, & F_A &= \frac{3}{n} \frac{1}{u^A} \sqrt{-v_0 (u^1 \dots u^n)^{\frac{3}{n}}}. \end{aligned} \tag{7.60}$$

The KK-scalar is given in terms of the harmonic functions (7.54) by

$$e^{-\phi} = \sqrt{-4\mathcal{H}_0(\mathcal{H}^1 \dots \mathcal{H}^3)^{\frac{3}{n}}},$$

and the integration constants must satisfy

$$-4h_0(h^1 \dots h^n)^{\frac{3}{n}} = 1.$$

As before, we can write v_0, u^A explicitly in terms of the harmonic functions using (7.56), and the original four-dimensional complex scalars $X^I = e^{-\frac{\phi}{2}} Y^I$ can be determined through (7.60) and the expression for the KK-scalar. The gauge fields can be determined through (7.42) or (7.16).

7.3.6 Field rotations and non-BPS solutions

In section 7.3.1 we presented a particular ansatz for finding extremal black holes. By explicitly computing the ADM mass and comparing it with the central charge it was shown that this ansatz always produces BPS solutions. However, since the BPS condition is stronger than the extremal condition we have not exhausted the search for all possible extremal solutions. In this section we will show that for models which allow for a ‘field rotation matrix’ we can modify the extremal ansatz to include a more general class of extremal solutions. This is similar to the case in five dimensions, as discussed in section 5.2. A field rotation matrix R^a_b is defined as any matrix that satisfies

$$\tilde{H}_{ab} R^a_c R^b_d = \tilde{H}_{cd}, \quad (7.61)$$

either on the whole para-quaternion Kähler target manifold or a totally geodesic submanifold. We call a field rotation matrix trivial if $R = \pm \mathbb{1}$. For models that allow for a non-trivial field rotation matrix one can generalise the second part of the extremal ansatz (7.38) to become

$$\partial_\mu q^a = R^a_b \partial_\mu \hat{q}^b. \quad (7.62)$$

Clearly for the trivial solution $R = \pm \mathbb{1}$ we recover our previous extremal ansatz. Solutions are again characterised by the harmonic functions (7.40), but they are now related to the scalar fields through

$$\partial_\mu q_a = R_a^b \mathcal{H}_b, \quad (7.63)$$

$$\partial_\mu \hat{q}^a = \tilde{H}^{ab} \partial_\mu \mathcal{H}_b, \quad (7.64)$$

where R_a^b is the transpose inverse of R^a_b , i.e. $R^a_b R_a^c = \delta_b^c$. From the above expressions we see that the field rotation matrix rotates the charges and asymptotic integration constants of the solution relative to the q^a scalar fields [104].

Let us investigate what effect the field rotation matrix has on the four-dimensional fields. If we first decompose R_a^b into blocks

$$R_a^b = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

then we can replace the expressions (7.16a) and (7.16b) with the more general expressions

$$\partial_\mu (e^{\phi/2} (X^J + \bar{X}^J)) A_J^I + \partial_\mu (e^{\phi/2} (F_J + \bar{F}_J)) C^{JI} = \pm (F_{0\mu}^{I+} + F_{0\mu}^{I-}), \quad (7.65a)$$

$$\partial_\mu (e^{\phi/2} (X^J + \bar{X}^J)) B_{JI} + \partial_\mu (e^{\phi/2} (F_J + \bar{F}_J)) D^J_I = \pm (G_{I|0\mu}^+ + G_{I|0\mu}^-). \quad (7.65b)$$

This confirms that the charges therefore appear rotated with respect to the scalar fields. Calculating the ADM mass of the solution we find that formula (7.46) gets modified in the expected fashion:

$$M_{\text{ADM}} = \frac{1}{4\pi} \oint_{S^2_\infty} d^2\Sigma^\mu \left| X^I \left(A_I{}^J \partial_\mu \hat{\mathcal{H}}_J + B_{IJ} \partial_\mu \hat{\mathcal{H}}^J \right) - F_I \left(C^{IJ} \partial_\mu \hat{\mathcal{H}}_J + D^I{}_J \partial_\mu \hat{\mathcal{H}}^J \right) \right|.$$

We conclude that introducing a non-trivial field rotation matrix into the solution will generically result in non-BPS solutions. Of course, special choices of parameters may exist for which one may still obtain a BPS solution.

A field rotation matrix is not guaranteed to exist for generic models, and therefore we cannot guarantee that non-BPS solutions exist for all models. However, for certain classes of models we can demonstrate the existence of a non-trivial field rotation matrix. It has been known for some time that one can systematically find field rotation matrices for models that are symmetric spaces [51, 52]. In work by the author it was shown that for axion-free configurations of models of the form $F = f(X^1, \dots, X^n)/X^0$, as described in section 7.3.3, a non-trivial field rotation matrix always exists [19]. An interesting feature of these solutions is that the field rotation matrix does not satisfy (7.61) on the whole para-quaternion Kähler manifold, but only on a totally geodesic submanifold. The non-BPS solutions therefore have half the dimensionality of the full BPS solutions. It is an open question as to how one can identify c-map spaces that allow for a field rotation matrix either on the whole space or on totally geodesic submanifolds.

One may also wonder whether a field rotation matrix allows us to find non-BPS rotating solutions. Unfortunately a field rotation matrix of the form (7.61) is not enough in this case as the Einstein equations are no longer solved generically. This suggests that in order to find non-BPS solutions one must not impose that the three-dimensional metric is flat. This makes sense because for rotating solutions the BPS bound corresponds to a flat three-dimensional metric, whereas the extremal bound corresponds to a curved three-dimensional metric. This means that deforming away from the BPS bound naturally requires a curved three-dimensional metric.

In the following section we will prove that a non-trivial field rotation matrix exists for the models considered in section 7.3.3, and thus find new non-BPS solutions to the whole class of models. We will then revisit the particular examples of the $STU + aU^3$ model and STU -like models and present new non-BPS solutions.

7.3.7 Non-BPS solutions of $F = \frac{f(Y^1, \dots, Y^n)}{Y^0}$ models

Let us revisit the class of models discussed in section 7.3.3, which were characterised by prepotentials of the form

$$F = \frac{f(Y^1, \dots, Y^n)}{Y^0}, \quad (7.66)$$

where f is the analytic continuation of a real function. We will show that for axion-free configurations a non-trivial field rotation matrix is guaranteed to exist. For simplicity we will focus on the specific case where $n = 3$, as this is relevant for models descending from the supergravity r-map, but one can extend the results to any $n \geq 1$ without loss of generality.

For this class of models the first part of the axion-free condition (7.49) ensures that the metric \tilde{H}_{ab} can be decomposed according to

$$\tilde{H}_{ab} = \left(\begin{array}{c|cccc|ccc} * & 0 & 0 & 0 & 0 & * & * & * \\ \hline 0 & * & * & * & * & 0 & 0 & 0 \\ 0 & * & * & * & * & 0 & 0 & 0 \\ 0 & * & * & * & * & 0 & 0 & 0 \\ 0 & * & * & * & * & 0 & 0 & 0 \\ \hline * & 0 & 0 & 0 & 0 & * & * & * \\ * & 0 & 0 & 0 & 0 & * & * & * \\ * & 0 & 0 & 0 & 0 & * & * & * \end{array} \right), \quad (7.67)$$

where a $*$ represents a possible non-zero entry. Let us show why this is true by considering the component \tilde{H}_{10} as an example. This component is given by

$$\tilde{H}_{10}\big|_{\sharp} = \left(\frac{\partial}{\partial x^0} \frac{\partial \tilde{H}}{\partial x^1} \right) \bigg|_{\sharp},$$

where \sharp denotes the restriction to axion-free configurations. Since the field x^0 does not appear in the axion-free condition it does not matter in which order we perform the partial derivative, and so we can write

$$\tilde{H}_{10}\big|_{\sharp} = \frac{\partial}{\partial x^0} \left(\frac{\partial \tilde{H}}{\partial x^1} \bigg|_{\sharp} \right).$$

Since the axion free condition implies $v_1 = 0$, which from (7.7) implies that $\frac{\partial \tilde{H}}{\partial x^1} \big|_{\sharp} = 0$, we conclude that

$$\tilde{H}_{10}\big|_{\sharp} = \frac{\partial}{\partial x^0}(0) = 0.$$

The same argument is true for any matrix element containing one index in $\{0, 5, 6, 7\}$ and one index in $\{1, 2, 3, 4\}$. Actually, the matrix \tilde{H}_{ab} decomposes even further. Using the formula for the Hesse potential (B.6) we observe that \tilde{H}_{ab} takes the more restrictive form

$$\tilde{H}_{ab} = \left(\begin{array}{c|cccc|cccc} \frac{1}{4(x^0)^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & * & * & * & * & 0 & 0 & 0 \\ 0 & * & * & * & * & 0 & 0 & 0 \\ 0 & * & * & * & * & 0 & 0 & 0 \\ 0 & * & * & * & * & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * & * \end{array} \right), \quad (7.68)$$

where the entries in the bottom-right block depend only on y_1, y_2, y_3 . The complete set of axion free conditions (7.49) and (7.50) ensure that the central block in the above expression completely decouples from the equations of motion, and will therefore be of no further relevance to the discussion.

We can conclude that after imposing that configurations are axion-free the metric \tilde{H}_{ab} admits a non-trivial field rotation matrix of the form

$$R^a_b = \pm \left(\begin{array}{c|c|c} -1 & 0 & 0 \\ \hline 0 & \pm \mathbb{1}_{n+1} & 0 \\ \hline 0 & 0 & \mathbb{1} \end{array} \right), \quad (7.69)$$

where the sign in the central block does not have any effect on the solution. We can therefore generically construct non-BPS solutions for this class of models.

7.3.8 Non-BPS solutions of the $STU + aU^3$ model

Since this model falls into the class of models considered above it admits the non-trivial field rotation matrix (7.69), and we can obtain non-BPS solutions. The harmonic functions for the non-extremal solutions are the same as before, and are given by (7.54). The KK-scalar differs from the BPS case (7.58) by a sign inside the square root

$$e^{-\phi} = \sqrt{4\mathcal{H}_0(\mathcal{H}^1\mathcal{H}^2\mathcal{H}^3 + a(\mathcal{H}^1)^3)}.$$

The asymptotic integration constants h^I, h_I satisfy the constraint

$$4h_0(h^1h^2h^3 + a(h^1)^3) = 1.$$

From the above expression we see clearly that in order to obtain the correct sign in the four-dimensional metric the integration constants h_0, h^A must be chosen differently from the BPS case. The values of the charges Q_0, P^A may also need to be chosen differently in order to avoid additional singularities in the metric. We can write v_0, u^A explicitly in terms of harmonic functions by

$$v_0 = \frac{1}{2}e^\phi\mathcal{H}_0, \quad u^A = -\frac{1}{2}e^\phi\mathcal{H}^A, \quad (7.70)$$

which differs from the BPS case (7.59) by the sign in front of the expression for v_0 . The original four-dimensional complex scalars $X^I = e^{-\frac{\phi}{2}}Y^I$ can then be determined through the expressions (7.57), which one can write explicitly in terms of harmonic functions using (7.70). The dependence of the gauge fields on the harmonic functions remains the same as for the BPS case.

7.3.9 Non-BPS solutions of STU -like models

The last class of models for which we will consider non-BPS solutions are STU -like models, as defined in section 7.3.5. Since these models also fall into the class of models considered in section 7.3.3 we can find a non-trivial field rotation matrix of the form given by (7.69). However, for these models we can actually find a more general field rotation matrix. This is because for these models the metric can be explicitly calculated

to be

$$\tilde{H}_{ab} = \left(\begin{array}{ccc|ccc} \frac{1}{4(x^0)^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & * & & * & 0 & 0 & 0 \\ 0 & & \ddots & & 0 & 0 & 0 \\ 0 & * & & * & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \frac{1}{4(y_1)^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4(y_n)^2} \end{array} \right). \quad (7.71)$$

This metric allows for a field rotation matrix of the form

$$R^a_b = \left(\begin{array}{c|cc} \pm 1 & 0 & 0 \\ 0 & \pm \mathbb{1}_{n+1} & 0 \\ \hline 0 & 0 & \begin{array}{ccc} \pm 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \pm 1 \end{array} \end{array} \right), \quad (7.72)$$

which is clearly more general than (7.69).

Non-BPS solutions are characterised by the harmonic functions (7.54), but now the KK-scalar takes the form

$$e^{-\phi} = \sqrt{\pm 4\mathcal{H}_0(\mathcal{H}^1 \dots \mathcal{H}^3)^{\frac{3}{n}}},$$

where the choice of sign depends on the particular sign choices in the field rotation matrix (7.72). The asymptotic integration constants h^I, h_I must satisfy the constraint

$$\pm 4h_0(h^1 \dots h^n)^{\frac{3}{n}} = 1.$$

This time it is possible to choose the same asymptotic integration constants as before and still obtain the correct metric signature in four dimensions. We can write v_0, u^A explicitly in terms of harmonic functions by

$$v_0 = \mp \frac{1}{2} e^{\phi} \mathcal{H}_0, \quad u^A = \mp \frac{1}{2} e^{\phi} \mathcal{H}^A, \quad (7.73)$$

where the sign depends on the sign choices of (7.72). As usual, the original four-dimensional scalar fields $X^I = e^{-\frac{\phi}{2}} Y^I$ can be determined using (7.60) and (7.73). The expression for the gauge fields does not change in the non-BPS case.

7.4 Static non-extremal black holes

From our experience of the Reissner–Nordström black hole in section 3.2.2, and the five-dimensional non-extremal solutions constructed in section 5.3, we no longer expect the three-dimensional part of the metric to be flat. Also following the lead of these examples we will impose that spacetime is spherically symmetric. We shall see that the requirement of spherical symmetry is enough to completely fix the three-dimensional part of the metric.

Unlike in the extremal case we cannot find non-extremal solutions for generic models. We will only consider axion-free solutions to models with prepotentials of the form

$F = f(X^1, \dots, X^n)/X^0$, as described in section 7.3.3. We will first consider the subclass of these models given by *STU*-like models, since in this case we can explicitly integrate the second order equations of motion and find the general solution. We then investigate the criteria for these solutions to describe non-extremal black holes with scalar fields that take finite values. We find that the number of integration constants must reduce by half. We then discuss generic non-extremal solutions to the whole class of models of the form $F = f(X^1, \dots, X^n)/X^0$, however we find that some scalar fields must be proportional to one another. Like in the five-dimensional case in section 5.3.2, we find that we can construct non-extremal solutions that have as many independent scalar fields as there are blocks in the metric. Since all metrics for these models can be split into at least two blocks (which is different to the models we considered in five dimensions) we obtain a universal non-extremal solution that has one non-constant physical scalar field. All non-extremal solutions presented in this section are new solutions that will appear in a later publication [61], with the exception of the *STU* model for which solutions were given in [104].

7.4.1 Non-extremal ansatz

We will now present an ansatz for finding non-extremal black hole solutions. This is similar to the extremal ansatz, the only difference being that the three-dimensional metric is no longer flat, but we do impose that it is spherically symmetric.

We begin by imposing that spacetime is static, in which case the KK-vector vanishes. In terms of the fields appearing in the Lagrangian (7.1) this corresponds to setting

$$\frac{1}{2H} \left(\partial_\mu \tilde{\phi} + 2\hat{q}^a \Omega_{ab} \partial_\mu \hat{q}^b \right) = 0 .$$

The first part of our non-extremal ansatz is the same as for the extremal ansatz, we set

$$q^a \Omega_{ab} \partial_\mu q^b = q^a \Omega_{ab} \partial_\mu \hat{q}^b = 0 , \quad (7.74)$$

which, as mentioned in section 7.3.1, is equivalent to imposing the second part of the isotropic ansatz of section 7.2.1 for static backgrounds.

We will now show that by imposing spherical symmetry, the three-dimensional part of the metric according to the decomposition (6.28) is always given by the three-dimensional part of the Reissner–Nordström solution. The derivation is entirely analogous to the five-dimensional case; see section 5.3. We start by imposing that spacetime is spherically symmetric. We will parametrise the radial direction by an affine parameter τ , which has the property that

$$\Delta_4 = \frac{d^2}{d\tau^2} + \text{terms independent of } \tau .$$

The three-dimensional part of a general static and spherically symmetric metric can be brought to the form [49]

$$ds_3^2 = e^{4A(\tau)} d\tau^2 + e^{2A(\tau)} d\Omega_{(2)}^2 .$$

Imposing that solutions are spherically symmetric implies that the LHS of the Einstein equations (7.5) where $\mu \neq \tau$ or $\nu \neq \tau$ must vanish. Since the RHS of these equations are proportional to $\ddot{A} - e^{2A}$, this implies that $\ddot{A} - e^{2A} = 0$. This can be integrated to give $\dot{A}^2 = c^2 + e^{2A}$, where we have chosen the integration constant c^2 to be positive in

order to avoid solutions that are periodic in the radial coordinate. We relabel $x = e^{-A}$ to write this as

$$\dot{x} = \pm \sqrt{c^2 x^2 + 1} ,$$

which is solved by

$$x(\tau) = \frac{\sinh(\pm c\tau + D)}{c} .$$

To ensure x is strictly positive in the range $0 < \tau < +\infty$ we choose the positive sign and $D = 0$, i.e.

$$e^{-A} = x(\tau) = \frac{\sinh(c\tau)}{c} .$$

The three-dimensional line element is then given by the three-dimensional part of the Reissner–Nordström metric

$$ds_3^2 = \frac{c^4}{\sinh^4 c\tau} d\tau^2 + \frac{c^2}{\sinh^2 c\tau} d\Omega_{(2)}^2 . \quad (7.75)$$

This is true more generally when reducing any four-dimensional sigma model over time [30]. The (τ, τ) component of the Ricci tensor corresponding to this metric is given by $R_{\tau\tau} = 2c$. Therefore, the equations of motion after imposing staticity, spherical symmetry and the first part of our non-extremal ansatz (7.74) are given simply by

$$\frac{d}{d\tau} \left(\tilde{H}_{ab} \dot{q}^b \right) - \frac{1}{2} \partial_a \tilde{H}_{bc} \left(\dot{q}^b \dot{q}^c - \dot{q}^b \dot{q}^c \right) = 0 , \quad (7.76)$$

$$\frac{d}{d\tau} \left(\tilde{H}_{ab} \dot{q}^b \right) = 0 , \quad (7.77)$$

$$\tilde{H}_{ab} \left(\dot{q}^a \dot{q}^b - \dot{q}^a \dot{q}^b \right) = c^2 . \quad (7.78)$$

The equations (7.76) and (7.77) follow from the variation of the one-dimensional effective action

$$\mathcal{L} = \tilde{H}_{ab} \left(\dot{q}^a \dot{q}^b - \dot{q}^a \dot{q}^b \right) ,$$

which must then be supplemented by (7.78) which acts as a Hamiltonian constraint.

At this stage it is convenient to switch to the dual coordinates q_a introduced in section 7.1. We shall also define $\hat{q}_a := \tilde{H}_{ab} \dot{q}^b$, which due to the equation of motion (7.77) does not place any additional integrability condition on the \hat{q}^a . The equations of motion now take the particularly simple form

$$\ddot{q}_a + \frac{1}{2} \partial_a \tilde{H}^{bc} \left(\dot{q}_b \dot{q}_c - \dot{q}_b \dot{q}_c \right) = 0 , \quad (7.79)$$

$$\ddot{\hat{q}}_a = 0 , \quad (7.80)$$

$$\tilde{H}^{ab} \left(\dot{q}_a \dot{q}_b - \dot{q}_a \dot{q}_b \right) = c^2 . \quad (7.81)$$

Notice the similarity with the analogous equations of motion in the five-dimensional case (5.17) – (5.19). The equations of motion for the \hat{q}^a fields can immediately be solved by setting

$$\tilde{H}_{ab} \dot{q}^b = \mathcal{Q}_a := \begin{pmatrix} -Q_I \\ P^I \end{pmatrix} , \quad (7.82)$$

where Q_I and P^I are the constant electric and magnetic charges carried by the solution. We are left to solve the equation of motion for the q^a coordinates along with the Einstein

equation

$$\ddot{q}_a + \frac{1}{2} \partial_a \tilde{H}^{bc} (\dot{q}_b \dot{q}_c - \mathcal{Q}_b \mathcal{Q}_c) = 0 , \quad (7.83)$$

$$\tilde{H}^{ab} (\dot{q}_a \dot{q}_b - \mathcal{Q}_a \mathcal{Q}_b) = c^2 . \quad (7.84)$$

At the time of writing there exists no universal ansatz that solves (7.83) and (7.84) for generic models, at least without setting all the scalar fields to be proportional to one another. We therefore have to consider solutions for different classes of models separately. In this thesis we will investigate axion-free solutions for two distinct classes of models, both of which have prepotentials of the $F = f(X^1, \dots, X^n)/X^0$ as described in section 7.3.3. First, we consider *STU*-like models (see section 7.3.5), where we integrate the full second order equations of motion directly and find the general solution. Second, we consider solutions to models that have block diagonal metrics, for which we find solutions with as many independent scalar fields as there are blocks in the metric. Since generic models of the form $F = f(X^1, \dots, X^n)/X^0$ always have at least two blocks we can always find a universal solution with one non-constant physical scalar field.

7.4.2 The *STU*-like models

For axion-free configurations of the generalised *STU* model we can actually find the general solution to the equations (7.83) and (7.84). The Hesse potential for such configurations is given in appendix B:

$$H(x, y) = -2 \left(y_0 (x^1 \dots x^n)^{\frac{3}{n}} \right)^{\frac{1}{2}} .$$

The equations of motion reduce to

$$\begin{aligned} \ddot{q}_0 - \frac{(\dot{q}_0^2 - \mathcal{Q}_0^2)}{q_0} &= 0 , \\ \ddot{q}_{n+1} - \frac{(\dot{q}_{n+1}^2 - \mathcal{Q}_{n+1}^2)}{q_{n+1}} &= 0 , \\ &\vdots \end{aligned} \quad (7.85)$$

$$\begin{aligned} \ddot{q}_{2n+1} - \frac{(\dot{q}_{2n+1}^2 - \mathcal{Q}_{2n+1}^2)}{q_{2n+1}} &= 0 , \\ \frac{(\dot{q}_0^2 - \mathcal{Q}_0^2)}{4q_0^2} + \frac{3(\dot{q}_{n+1}^2 - \mathcal{Q}_{n+1}^2)}{4nq_{n+1}^2} + \dots + \frac{3(\dot{q}_{2n+1}^2 - \mathcal{Q}_{2n+1}^2)}{4nq_{2n+1}^2} &= c^2 . \end{aligned} \quad (7.86)$$

The second order equations of motion (7.85) completely decouple from one another and can be explicitly integrated to obtain the general solution

$$\begin{aligned} q_0 &= -\pm \frac{Q_0}{B_0} \sinh \left(B_0 \tau + B_0 \frac{h_0}{Q_0} \right) , \\ q_{n+2} &= \pm \frac{P^1}{B^1} \sinh \left(B^1 \tau + B^1 \frac{h^1}{P^1} \right) , \\ &\vdots \\ q_{2n+2} &= \pm \frac{P^n}{B^n} \sinh \left(B^n \tau + B^n \frac{h^n}{P^n} \right) , \end{aligned} \quad (7.87)$$

where h_0, h^A, B_0, B^A are $(2n+2)$ independent integration constants. Since the solution is invariant under $B_0, B^A \rightarrow -B_0, -B^A$ we can assume without loss of generality that the B_0, B^A integration constants are non-negative. Choosing the signs in the above expressions has an effect on the extremal limit: choosing the same sign everywhere gives BPS black holes, whereas different signs will give non-BPS black holes. For convenience let us choose the positive sign in the all of the above expressions from now on. Plugging these solutions into the Einstein equation (7.86) gives an algebraic relation that determines the non-extremality parameter in terms of the B_0, B^A integration constants

$$\frac{1}{4}(B_0)^2 + \frac{3}{4n}(B^1)^2 + \dots + \frac{3}{4n}(B^n)^2 = c^2. \quad (7.88)$$

Let us now lift the solution described by (7.87) and (7.88) back up to four dimensions. Since the KK-scalar is related to the Hesse potential through $e^\phi = -2H$ we can determine this to be

$$e^{-\phi} = 4\sqrt{q_0 (q_{n+1} \dots q_{2n+1})^{\frac{3}{n}}}.$$

Using (6.28) the four-dimensional metric is determined to be

$$ds_4^2 = e^\phi dt^2 + e^{-\phi} \left(\frac{c^4}{\sinh^4 c\tau} d\tau^2 + \frac{c^2}{\sinh^2 c\tau} d\Omega_{(2)}^2 \right),$$

where

$$e^{-\phi} = \sqrt{4q_0 (q_{n+1} \dots q_{2n+1})^{\frac{3}{n}}}.$$

The area A of the outer horizon located at $\tau \rightarrow +\infty$ is given by

$$A = \lim_{\tau \rightarrow +\infty} \sqrt{4q_0 (q_{n+1} \dots q_{2n+1})^{\frac{3}{n}}} \frac{c^2}{\sinh^2 c\tau}.$$

The highest term in the numerator is proportional to

$$e^{(\frac{1}{2}B_0 + \frac{3}{2n}B^1 + \dots + \frac{3}{2n}B^n)\tau}.$$

In order to have a non-vanishing area this must match precisely with the highest order term in the denominator, given by $e^{2c\tau}$. We therefore have the expression

$$\frac{1}{4}B_0 + \frac{3}{4n}B^1 + \dots + \frac{3}{4n}B^n = c. \quad (7.89)$$

The scalar fields Y^I are given by (7.60). Recall that the physical scalars z^A are given by

$$z^A := \frac{X^A}{X^0} = \frac{Y^A}{Y^0} = iu^A \sqrt{\frac{-v_0}{(u^1 \dots u^n)^{\frac{3}{n}}}}.$$

For these physical scalars to take on finite values at the horizon we must set

$$B_0 = B^1 = \dots = B^n,$$

which combined with (7.89) means that

$$B_0 = B^1 = \dots = B^n = c.$$

In addition, in order to obtain Minkowski space at radial infinity we need to ensure that $e^\phi \rightarrow 1$, which places one more constraint on the integration constants

$$-4 \frac{Q_0}{c} \sinh \left(c \frac{h_0}{Q_0} \right) \left[\frac{P^1}{c} \sinh \left(c \frac{h^1}{P^1} \right) \dots \frac{P^n}{c} \sinh \left(c \frac{h^n}{P^n} \right) \right]^{\frac{3}{n}} = 1.$$

The number of integration constants therefore reduces from $(2n + 2)$ to n . This is precisely half what one would expect from solving the second order equations of motion for the n physical scalar fields z^A . This suggests that the solution actually solves first order equations.

Now let us consider the scalar fields. In terms of the dual scalar fields the solution is written as

$$q_0 = -\frac{\mathcal{H}_0}{\sqrt{W}}, \quad q_{n+2} = \frac{\mathcal{H}^1}{\sqrt{W}}, \quad \dots \quad q_{2n+2} = \frac{\mathcal{H}^n}{\sqrt{W}},$$

where $\mathcal{H}_0, \mathcal{H}^A$ and W are harmonic functions with respect to the flat metric on \mathbb{R}^3 , and can be written in terms of either τ or r by

$$\begin{aligned} \mathcal{H}_0 &= \frac{Q_0}{2c} e^{c \frac{h_0}{Q_0}} - \frac{Q_0}{2c} e^{-2c\tau - c \frac{h_0}{Q_0}} &= \frac{Q_0}{c} \sinh \left(c \frac{h_0}{Q_0} \right) - \frac{Q_0 e^{-c \frac{h_0}{Q_0}}}{r}, \\ \mathcal{H}^A &= \frac{P^A}{2c} e^{c \frac{h^A}{P^A}} - \frac{P^A}{2c} e^{-2c\tau - c \frac{h^A}{P^A}} &= \frac{P^A}{c} \sinh \left(c \frac{h^A}{P^A} \right) - \frac{P^A e^{-c \frac{h^A}{P^A}}}{r}, \\ W &= e^{-2c\tau} &= 1 - \frac{2c}{r}. \end{aligned}$$

We can write v_0, u^A in terms of harmonic functions as

$$v_0 = -\frac{1}{2} e^\phi \frac{\mathcal{H}_0}{\sqrt{W}}, \quad u^A = -\frac{1}{2} e^\phi \frac{\mathcal{H}^A}{\sqrt{W}},$$

and the original complex scalar fields $X^I = e^{-\frac{\phi}{2}} Y^I$ can be obtained through (7.60). The physical scalar fields are given by

$$z^A = i \mathcal{H}^A \sqrt{\frac{-\mathcal{H}_0}{(\mathcal{H}^1 \dots \mathcal{H}^n)^{\frac{3}{n}}}}.$$

The integration constants h_0, h^A must satisfy

$$-4 \frac{Q_0}{c} \sinh \left(c \frac{h_0}{Q_0} \right) \left[\frac{P^1}{c} \sinh \left(c \frac{h^1}{P^1} \right) \dots \frac{P^n}{c} \sinh \left(c \frac{h^n}{P^n} \right) \right]^{\frac{3}{n}} = 1.$$

The metric can now be written as

$$ds_4^2 = -\frac{W}{\sqrt{-4\mathcal{H}_0(\mathcal{H}^1 \dots \mathcal{H}^n)^{\frac{3}{n}}}} dt^2 + \sqrt{-4\mathcal{H}_0(\mathcal{H}^1 \dots \mathcal{H}^n)^{\frac{3}{n}}} \left(\frac{dr^2}{W} + r^2 d\Omega_{(2)}^2 \right).$$

The gauge fields are, as in the extremal case, given by the expressions (7.16a) and (7.16b). The above metric is a direct generalisation of the non-extremal Reissner–Nordström solution (3.13). Clearly if we set the harmonic functions $\mathcal{H}_0, \mathcal{H}^A$ proportional to one another we recover the Reissner–Nordström solution.

Note that by taking the limit $c \rightarrow 0$ and we find precisely the extremal solutions, either BPS or non-BPS, obtained in sections 7.3.5 and 7.3.9, which themselves generalise the BPS solution to the STU model given by (3.18).

7.4.3 Block diagonal models

For axion-free configurations of any model of the form $F = f(X^1, \dots, X^n)/X^0$ we may classify the $2(n+1) \times 2(n+1)$ metric \tilde{H}_{ab} according to the number of blocks appearing in the bottom right $n \times n$ entries. For example, consider an 8×8 metric of the form

$$\tilde{H}_{ab} = \left(\begin{array}{c|cccc|cccc} \frac{1}{4(x^0)^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & * & * & * & * & 0 & 0 & 0 \\ 0 & * & * & * & * & 0 & 0 & 0 \\ 0 & * & * & * & * & 0 & 0 & 0 \\ 0 & * & * & * & * & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \end{array} \right). \quad (7.90)$$

This metric has two blocks in the bottom-right 3×3 entries. We may perform row operations on the metric in order to write it using as small blocks in the bottom-right as possible, which corresponds to taking general linear combinations of the special real coordinates. Recall that by imposing the axion-free conditions (7.49) and (7.51) all entries in the central block completely decouple from the equations of motion. Generically metrics for axion-free configurations of models of the form $F = f(X^1, \dots, X^n)/X^0$ will only contain one bottom-right block. However one can construct certain classes that have smaller blocks. For example a model of the form

$$F = \frac{f_1(X^1, X^2)f_2(X^3)}{X^0},$$

will have a corresponding metric with two bottom-right blocks, as considered above: one of size 2×2 and another of size 1×1 .

We will now present a solution where we set all scalar fields within each bottom-right block proportional to one another. For concreteness we will suppose that the bottom-right entries split into just two blocks of size $k \times k$ and $l \times l$ where $k \geq 1$ and $l = n - k$. We now set the scalar fields within each block proportional to one another

$$y_1 \propto \dots \propto y_k, \quad y_{k+1} \propto \dots \propto y_n, \quad (7.91)$$

which implies that the v_A are proportional to each other within each block

$$u^1 \propto \dots \propto u^k, \quad u^{k+1} \propto \dots \propto u^n.$$

This means that the Hesse potential splits into two distinct factors

$$H = -4\sqrt{v_0 f_1(u^1, \dots, u^k) f_2(u^{k+1} \dots u^n)}.$$

Let us define the two independent dual scalar fields $q_{(1)}$ and $q_{(2)}$ by

$$q_{(1)} := \frac{u^1}{H}, \quad q_{(2)} := \frac{u^{k+1}}{H}.$$

Here we have labelled the two independent scalar fields as $q_{(1)}$ and $q_{(2)}$. We shall also fix the ratio between magnetic charges in each block to be

$$P^{(1)} := P^1 = \frac{u^1}{u^2} P^2 = \dots = \frac{u^1}{u^k} P^k, \quad (7.92)$$

$$P^{(2)} := P^{k+1} = \frac{u^{k+1}}{u^{k+2}} P^{k+2} = \dots = \frac{u^{k+1}}{u^n} P^n. \quad (7.93)$$

The equations of motion now reduce to

$$\ddot{q}_0 + \frac{[\dot{q}_0^2 - Q_0^2]}{\dot{q}_0} = 0 , \quad (7.94)$$

$$\ddot{q}_{(1)} + \frac{[\dot{q}_{(1)}^2 - P^{(1)2}]}{\dot{q}_{(1)}} = 0 , \quad (7.95)$$

$$\ddot{q}_{(2)} + \frac{[\dot{q}_{(2)}^2 - P^{(2)2}]}{\dot{q}_{(2)}} = 0 , \quad (7.96)$$

$$\frac{[\dot{q}_0^2 - Q_0^2]}{\dot{q}_0^2} + \psi_1 \frac{[\dot{q}_{(1)}^2 - P^{(1)2}]}{\dot{q}_{(1)}^2} + \psi_2 \frac{[\dot{q}_{(2)}^2 - P^{(2)2}]}{\dot{q}_{(2)}^2} = c^2 , \quad (7.97)$$

where ψ_1 and ψ_2 are constants that depend on the particular choice of proportionality factors in (7.91), and must satisfy $\psi_1 + \psi_2 = 0$. The second order equations (7.94), (7.95) and (7.96) can be solved immediately to give

$$q_0 = \pm \frac{Q_0}{B_0} \sinh \left(B_0 \tau + B_0 \frac{h_0}{Q_0} \right) , \quad (7.98)$$

$$q_{(1)} = \pm \frac{P^{(1)}}{B^{(1)}} \sinh \left(B^{(1)} \tau + B^{(1)} \frac{h^{(1)}}{P^{(1)}} \right) , \quad (7.99)$$

$$q_{(2)} = \pm \frac{P^{(2)}}{B^{(2)}} \sinh \left(B^{(2)} \tau + B^{(2)} \frac{h^{(2)}}{P^{(2)}} \right) . \quad (7.100)$$

The Einstein equations (7.97) determine c in terms of the integration constants

$$Q_0 + \psi_1 P^{(1)} + \psi_2 P^{(2)} = c^2 .$$

The physical scalar fields z^A are given by

$$z^{(1)} := i \frac{q_0}{q_{(1)}} = z^1 = \frac{u_1}{u_2} z^2 = \dots = \frac{u_1}{u_k} z^k , \quad (7.101)$$

$$z^{(2)} := i \frac{q_0}{q_{(2)}} = z^{k+1} = \frac{u_{k+1}}{u_{k+2}} z^{k+2} = \dots = \frac{u_{k+1}}{u_n} z^n . \quad (7.102)$$

Let us now lift this solution back up to four dimensions. The KK scalar is given by

$$e^\phi = \mu q_0 (q_{(1)})^{\frac{k}{n}} (q_{(2)})^{\frac{l}{n}} ,$$

where μ is a constant which depends on the factors in (7.91). The four-dimensional metric is therefore given by

$$ds_4^2 = - \frac{1}{\mu q_0 (q_{(1)})^{\frac{k}{n}} (q_{(2)})^{\frac{l}{n}}} dt^2 + \mu q_0 (q_{(1)})^{\frac{k}{n}} (q_{(2)})^{\frac{l}{n}} \left(\frac{c^4}{\sinh^4 c\tau} d\tau^2 + \frac{c^2}{\sinh^2 c\tau} d\Omega_{(2)}^2 \right) .$$

The area A of the outer horizon located at $\tau \rightarrow +\infty$ is given by

$$A = \lim_{\tau \rightarrow +\infty} \psi q_0 (q_{(1)})^{\frac{k}{n}} (q_{(2)})^{\frac{l}{n}} \frac{c^2}{\sinh^2 c\tau} .$$

The highest term in the numerator is proportional to $\exp((B_0 + \frac{k}{n}B^{(1)} + \frac{l}{n}B^{(2)})\tau)$. In order to have a non-vanishing area this must match precisely with the highest order term in the denominator, given by $\exp(2c\tau)$. We therefore have the expression

$$B_0 + \frac{k}{n}B^{(1)} + \frac{l}{n}B^{(2)} = c. \quad (7.103)$$

The physical scalar fields are given by (7.101) and (7.102). In order for these physical scalar fields to be smooth on the horizon it is clear that we need

$$B_0 = B^{(1)} = B^{(2)},$$

which combined with (7.103) gives

$$B_0 = B^{(1)} = B^{(2)} = c.$$

In order for $e^\phi \rightarrow 1$ at radial infinity we also need

$$\psi \frac{Q_0}{c} \sinh\left(c \frac{h_0}{Q_0}\right) \left(\frac{P^{(1)}}{c} \sinh\left(c \frac{h^{(1)}}{P^{(1)}}\right)\right)^{\frac{k}{n}} \left(\frac{P^{(2)}}{c} \sinh\left(c \frac{h^{(2)}}{P^{(2)}}\right)\right)^{\frac{l}{n}} = 1.$$

Collecting everything together we can write the full solution to a metric with two blocks in the bottom-right as

$$ds_4^2 = -\frac{W}{\mathcal{H}_0(\mathcal{H}^{(1)})^{\frac{k}{n}}(\mathcal{H}^{(2)})^{\frac{l}{n}}} dt^2 + \mathcal{H}_0(\mathcal{H}^{(1)})^{\frac{k}{n}}(\mathcal{H}^{(2)})^{\frac{l}{n}} \left(\frac{dr^2}{W} + r^2 d\Omega_{(2)}^2\right),$$

where $\mathcal{H}_0, \mathcal{H}^{(1)}, \mathcal{H}^{(2)}$ and W are given by

$$\begin{aligned} \mathcal{H}_0 &= \frac{Q_0}{2c} e^{c \frac{h_0}{Q_0}} - \frac{Q_0}{2c} e^{-2c\tau - c \frac{h_0}{Q_0}} = \frac{Q_0}{c} \sinh\left(c \frac{h_0}{Q_0}\right) - \frac{Q_0 e^{-c \frac{h_0}{Q_0}}}{r}, \\ \mathcal{H}^{(1)} &= \frac{1}{\mu} \left[\frac{P^{(1)}}{2c} e^{c \frac{h^{(1)}}{P^{(1)}}} - \frac{P^{(1)}}{2c} e^{-2c\tau - c \frac{h^{(1)}}{P^{(1)}}} \right] = \frac{1}{\mu} \left[\frac{P^{(1)}}{c} \sinh\left(c \frac{h^{(1)}}{P^{(1)}}\right) - \frac{P^{(1)} e^{-c \frac{h^{(1)}}{P^{(1)}}}}{r} \right], \\ \mathcal{H}^{(2)} &= \frac{1}{\mu} \left[\frac{P^{(2)}}{2c} e^{c \frac{h^{(2)}}{P^{(2)}}} - \frac{P^{(2)}}{2c} e^{-2c\tau - c \frac{h^{(2)}}{P^{(2)}}} \right] = \frac{1}{\mu} \left[\frac{P^{(2)}}{c} \sinh\left(c \frac{h^{(2)}}{P^{(2)}}\right) - \frac{P^{(2)} e^{-c \frac{h^{(2)}}{P^{(2)}}}}{r} \right], \\ W &= e^{-2c\tau} = 1 - \frac{2c}{r}. \end{aligned}$$

The scalar fields are given by

$$q_0 = -\frac{\mathcal{H}_0}{\sqrt{W}}, \quad q_{(1)} = \frac{\mathcal{H}^{(1)}}{\sqrt{W}}, \quad q_{(2)} = \frac{\mathcal{H}^{(2)}}{\sqrt{W}},$$

and the physical scalars by

$$z^{(1)} = \sqrt{\frac{-\mathcal{H}_0}{\mathcal{H}^{(1)}}}, \quad z^{(2)} = \sqrt{\frac{-\mathcal{H}_0}{\mathcal{H}^{(2)}}}.$$

Since every model of the form $F = f(X^1, \dots, X^n)/X^0$ has at least one block in the bottom-right of the metric, we can give a universal solution for this class of models.

This solution will be characterised by a single electric and magnetic charge, and one non-constant scalar field. The full solution is given by

$$ds_4^2 = \frac{W}{\mathcal{H}_0 \mathcal{H}^1} dt^2 + \mathcal{H}_0 \mathcal{H}^1 \left(\frac{dr^2}{W} + r^2 d\Omega_{(2)}^2 \right) .$$

where $\mathcal{H}_0, \mathcal{H}^1$ and W are given by

$$\begin{aligned} \mathcal{H}_0 &= \frac{Q}{2c} e^{c \frac{h_0}{Q}} - \frac{Q}{2c} e^{-2c\tau - c \frac{h_0}{Q}} = \frac{Q}{c} \sinh \left(c \frac{h_0}{Q} \right) - \frac{Q e^{-c \frac{h_0}{Q}}}{r} , \\ \mathcal{H}^1 &= \frac{P}{2c} e^{c \frac{h^1}{P}} - \frac{P}{2c} e^{-2c\tau - c \frac{h^1}{P}} = \frac{P}{c} \sinh \left(c \frac{h^1}{P} \right) - \frac{P e^{-c \frac{h^1}{P}}}{r} , \\ W &= e^{-2c\tau} = 1 - \frac{2c}{r} . \end{aligned}$$

The one non-constant physical scalar field is given by $z^{(1)} = \sqrt{\frac{-\mathcal{H}_0}{\mathcal{H}^1}}$.

Chapter 8

Conclusion and outlook

In this thesis we developed our understanding of the r-map and c-map from a geometrical point of view, and showed how both maps can be used to generate new stationary solutions for theories of $\mathcal{N} = 2$ supergravity coupled to vector multiplets in five and four dimensions.

For the r-map we found that we could actually consider a more general theory than $5d, \mathcal{N} = 2$ supergravity by relaxing the condition that the target manifold is a projective special real manifold and only requiring that it is a generalised projective special real manifold. This was characterised by the fact that the Hesse potential on the corresponding d-conic Hessian manifold was only required to be a homogeneous function rather than a homogeneous polynomial of degree three. We then reduced this theory over a single spacelike or timelike dimension, and showed that for the supergravity r-map the target manifold geometry is projective special Kähler or projective special para-Kähler respectively.

We then investigated how the generalised r-map could be used to find extremal solutions, and how these solutions could be deformed into new non-extremal solutions. This deformation was obtained in a systematic way, and at every stage of the analysis we could switch off the non-extremality parameter and recover the known extremal solutions. We then investigated various properties of these non-extremal solutions. One novel feature was that in order to obtain black hole solutions with scalar fields that take finite values we needed the number of independent integration constants to halve. This suggests that non-extremal black holes always satisfy first-order equations, similar to their extremal counterparts. We also saw that both the generalised stabilisation equations and the black hole attractor equations can be directly generalised to the non-extremal case. It would be interesting to investigate the physical interpretation of these equations in the future.

The treatment of the r-map and investigation of black hole solutions lead us naturally to consider a similar approach toward the c-map, which is the more complicated of the two. In order to make manifest the similarities between the r-map and c-map, and also as an interesting study in its own right, we presented a new formulation of projective special Kähler geometry based on real coordinates. One key feature was that we worked at the level of the larger conic affine special Kähler manifold, and considered horizontal fields that could be projected down to the projective special Kähler manifold. A major practical advantage is that full symplectic covariance can be kept manifest at the level of the Lagrangian.

We then considered the c-map by reducing a theory of $4d, \mathcal{N} = 2$ supergravity coupled to vector multiplets over a single spacelike or timelike dimension. After du-

alising vector fields into scalar fields one is left with a three-dimensional Lagrangian describing gravity coupled to a non-linear sigma model. We then used the real formulation of projective special Kähler geometry to formulate this sigma model purely in terms of real fields, which gave us a new formulation of the c-map. We were easily able to use this to recover the result of Ferrara and Sabharwal that the target manifold is quaternion Kähler when reduction is performed over a spacelike dimension. We went on to prove the new result that the target manifold is para-quaternion Kähler when reduction is performed over a timelike dimension. We also showed that the spacelike c-map contained an additional integrable complex structure, separate from the known integrable complex structure and the quaternion structure. The timelike c-map contains two integrable complex structures, where again one integrable complex structure is distinct from the para-quaternion structure.

As an application of the new formulation of the c-map we investigated new stationary solutions for the four-dimensional theory. Since the c-map in real coordinates takes a very similar form to the r-map we were able to apply many of the same techniques we had previously developed. We began by discussing rotating solutions, but unfortunately they always contain a naked singularity and therefore cannot be classed as black holes. These solutions corresponded to identifying totally geodesic and totally isotropic submanifolds of the para-quaternion Kähler target manifold, and therefore produce solutions with a flat three-dimensional part of the metric. We leave the investigation of more complicated rotating solutions to future work.

After discussing rotating solutions we went on to investigate static solutions, for which the equations of motion simplify considerably, and are extremely similar to the equations of motion for the five-dimensional theory. We recovered the known extremal BPS solutions and went on to construct new non-BPS solutions for a large class of models, which contained, in particular, all models in the image of the r-map. Finally, we considered non-extremal black hole solutions. For *STU*-like models we could integrate the second order equations of motion directly and find the most general solution. As in the five-dimensional case, we found that for these solutions to correspond to non-extremal black holes in four dimensions with finite scalar fields the number of integration constants must halve. This suggests that the description of non-extremal black holes in terms of first order equations is universal and independent of the number of dimensions. We then went on to provide non-extremal solutions for generic models, where the number of independent scalar fields is proportional to the number of blocks in the metric. Since all metrics that we consider have at least two blocks we find a universal non-extremal solution with one non-constant scalar field.

Let us now discuss some areas that are open to future work. The most obvious extension of this work is to construct new solitonic supergravity solutions, other than the new non-extremal solutions that will appear shortly in [61]. We have seen that our formalism makes extensive use of the Hesse potential explicitly, and one limiting factor at present is the lack of explicit examples of Hesse potentials. We will come back to this point shortly. However, even for the *STU* model there is a realistic possibility of finding new rotating solutions with non-constant scalar fields, which includes both extremal non-BPS and non-extremal solutions. Another possibility is to look for solutions for different theories. For example, the formalism presented here has already been used to construct new non-BPS and non-extremal black hole solutions for various models in Fayet-Iliopoulos gauged supergravity [20, 21]. It might also be interesting to look more closely at how instantons are related to the solutions based in this thesis. By starting in an Euclidean theory in four dimensions and reducing over space one may be able to

construct new instantonic solutions of supergravity. The details of this reduction have already been calculated and will appear shortly [28].

On the geometrical side it would be interesting to investigate further the implications of the real formulation of special geometry presented in this thesis. One concrete goal would be to establish a relationship between the triple derivatives of the Hesse potential and the Riemannian curvature tensor, in a manner similar to the triple derivatives of the prepotential [77]. On a related topic, it would also be useful to both mathematicians and physicists to provide a dictionary between Hesse potentials and homogeneous spaces in a manner similar to that which exists for homogeneous spaces and the prepotential [24]. Another open question is whether projective special Kähler geometry can be generalised in a similar way to projective special real geometry. This would in turn open up the possibility of generalising the c-map, which may lead to new examples of (para-)quaternion Kähler manifolds or generalisations thereof.

Appendix A

Non-linear sigma models

A non-linear sigma model is quite simply a Lagrangian of a set of scalar fields Φ^i that takes the form

$$\mathbf{e}_d^{-1} \mathcal{L}_d = G_{ij}(\Phi) \partial_\mu \Phi^i \partial^\mu \Phi^j .$$

The terminology ‘non-linear’ refers to the fact that the couplings $G_{ij}(\Phi)$ need not be linear in Φ , while ‘sigma model’ simply refers to the label of the scalar fields in the first paper in which such Lagrangians were defined [112]. We can interpret the couplings $G_{ij}(\Phi)$ as a metric on some target manifold. Solutions are given by

$$\Delta \Phi^i + \Gamma_{jk}^i(\Phi) \frac{\partial \Phi^j}{\partial x^\mu} \frac{\partial \Phi^k}{\partial x^\nu} g^{\mu\nu} = 0 ,$$

and are called harmonic maps. Note that here $\Gamma_{jk}^i(\Phi)$ are the components of the Levi-Civita connection on the target manifold.

Harmonic maps also appear in the mathematics literature (see [113] for a useful review), which we will now briefly discuss. Our aim will be to understand harmonic maps in a geometrical way through the energy functional and tension field. Consider a differentiable map Φ between two Riemannian manifolds (M, g) and (N, G)

$$\Phi : (M, g) \longmapsto (N, G) .$$

The differential of Φ at a point $p \in M$ is given by

$$d\Phi : M \longrightarrow T^*M \otimes \Phi^{-1}TN ,$$

where $\Phi^{-1}TN$ is the pull-back vector bundle over M with fibres $T_{\Phi(p)}N$ at $p \in M$. Consider a coordinate patch $U \subset M$ and $V \subset N$ with $\Phi(U) \subset V$, with coordinates (x^1, \dots, x^m) on U and $\Phi(x) = (y^1, \dots, y^n)$ on V . We can then write the differential of Φ at the point $p \in M$ as

$$d\Phi_p = \frac{\partial \Phi^i}{\partial x^\mu}(p) dx_p^\mu \otimes \left(\frac{\partial}{\partial y^i} \right)_{\Phi(p)} \in \Gamma(T_p^*M \otimes T_{\Phi(p)}N) .$$

The inner product on $T^*M \otimes \Phi^{-1}TN$ is defined as

$$\langle \alpha \otimes A, \beta \otimes B \rangle = g^{-1}(\alpha, \beta) G(A, B) , \quad \text{where } \alpha, \beta \in T^*M ; A, B \in \Phi^{-1}TN ,$$

and we can then define the energy functional

$$e(\Phi)(x) = \frac{1}{2} |d\Phi(x)|^2 = \frac{1}{2} g^{\mu\nu} G_{ij} \frac{\partial \Phi^i}{\partial x^\mu} \frac{\partial \Phi^j}{\partial x^\nu} .$$

The energy of the map $\Phi : M \rightarrow N$ is then defined as

$$E(\Phi) = \int_M d\text{vol}_g \, e(\Phi) .$$

A map is harmonic if it is an extremum of the energy functional. One can then interpret a harmonic map as follows: The map Φ induces a natural vector field $\tau(\Phi) \in \Gamma(\Phi^{-1}TN)$ through

$$\tau(\Phi) = \text{Tr } \nabla d\Phi = \underbrace{(g^{\mu\nu} \nabla_\mu \nabla_\nu \Phi^i)}_{\tau^i(\Phi)} \frac{\partial}{\partial y^i} \circ \Phi .$$

This is called the tension field, and represents the direction in which the energy decreases most rapidly. We can write the components $\tau^i(\Phi)$ as

$$\begin{aligned} \tau^i(\Phi) &= g^{\mu\nu} \left(\frac{\partial^2 \Phi^i}{\partial x^\mu \partial x^\nu} - {}^M\Gamma_{\mu\nu}^\rho(x) \frac{\partial \Phi^i}{\partial x^\rho} + {}^N\Gamma_{jk}^i(\Phi) \frac{\partial \Phi^j}{\partial x^\mu} \frac{\partial \Phi^k}{\partial x^\nu} \right) \\ &= \Delta \Phi^i + {}^N\Gamma_{jk}^i(\Phi) \frac{\partial \Phi^j}{\partial x^\mu} \frac{\partial \Phi^k}{\partial x^\nu} g^{\mu\nu} , \end{aligned}$$

and we can see that the tension field vanishes identically iff the map Φ is harmonic.

Appendix B

Hesse potentials

In this section we will present the Hesse potentials for various models. We will not go through the details of each derivation, as these all appear in existing publications by the author. As in the main text, we define

$$x^I + iu^I := Y^I, \quad y_I + iv_I := F_I.$$

The $F = -iY^0Y^1$ model [20]:

$$H = -2(x^0x^1 + y_0y_1). \quad (\text{B.1})$$

The $F = -\frac{(Y^1)^3}{Y^0}$ model [20]:

$$H = -2\left(-y_0x^0y_0x^0 - 2y_0x^0y_1x^1 + \frac{1}{3}y_1x^1y_1x^1 + \frac{4}{27}x^0(y_1)^3 - 4y_0(x^1)^3\right)^{1/2}. \quad (\text{B.2})$$

The $F = -2i\sqrt{Y^0Y^1Y^2Y^3}$ model [21]:

$$H = -2\left(-(y_0x^0 - y_1x^1 - y_2x^2 - y_3x^3)^2 + 4y_1x^1y_2x^2 + 4y_1x^1y_3x^3 + 4y_2x^2y_3x^3 + 4y_0y_1y_2y_3 + 4x^0x^1x^2x^3\right)^{1/2}. \quad (\text{B.3})$$

The $F = -\frac{Y^1Y^2Y^3}{Y^0}$ model [19]:

$$H = -2\left(-(y.x)^2 + 4y_1x^1y_2x^2 + 4y_1x^1y_3x^3 + 4y_2x^2y_3x^3 + 4x^0y_1y_2y_3 - 4y_0x^1x^2x^3\right)^{1/2}. \quad (\text{B.4})$$

Note that (B.3) and (B.4) are related by a symplectic transformation. Since the Hesse potential is invariant under an overall U(1) phase transformation of Y^I and F_I , these Hesse potentials can alternatively be expressed in terms of u^I, v_I by simply making the substitutions $x^I \leftrightarrow u^I$ and $y_I \leftrightarrow v_I$.

Let us now consider models of the form

$$F = \frac{f(Y^1, \dots, Y^n)}{Y^0}, \quad (\text{B.5})$$

where f is real when evaluated on real fields. Note that since F is homogeneous of degree two we must have f homogeneous of degree three. In this case it is not necessarily possible to calculate the Hesse potential explicitly, but for the submanifold corresponding to Y^0 purely real and Y^A purely imaginary one can show that the Hesse potential takes the form [19]

$$H = -4\sqrt{x^0 h(y_1, \dots, y_n)}, \quad (\text{B.6})$$

where h is some specific but undetermined homogeneous function of degree three. The Hesse potential can be expressed in terms of u^I and v_I by

$$H = -4\sqrt{v_0 f(u^1, \dots, u^n)}, \quad (\text{B.7})$$

where f is the same function that appears in the prepotential. Note that the $U(1)$ symmetry corresponding to the overall phase of Y^I and F_I is explicitly broken by the requirement that Y^0 is real and Y^A are imaginary.

An important class of models of the form (B.5) is given by STU -like models, which are characterised by prepotentials of the form

$$F = -\frac{(Y^1 \dots Y^n)^{\frac{3}{n}}}{Y^0}.$$

For configurations of such models where Y^0 is purely real and Y^A purely imaginary we may calculate the function h appearing in (B.6) explicitly, and so we can give the Hesse potential explicitly in terms of x^I, y_I :

$$H = -4\sqrt{-x^0 (y_1, \dots, y_n)^{\frac{3}{n}}}. \quad (\text{B.8})$$

From (B.7) we can write the Hesse potential in terms of u^I, v_I as

$$H = -4\sqrt{-v_0 (u^1, \dots, u^n)^{\frac{3}{n}}}. \quad (\text{B.9})$$

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